Combinatorial R matrices for a family of crystals : $C_n^{(1)}$ and $A_{2n-1}^{(2)}$ cases

Goro Hatayama,* Atsuo Kuniba,*
Masato Okado† and Taichiro Takagi‡

Abstract

The combinatorial R matrices are obtained for a family $\{B_l\}$ of crystals for $U'_q(C_n^{(1)})$ and $U'_q(A_{2n-1}^{(2)})$, where B_l is the crystal of the irreducible module corresponding to the one-row Young diagram of length l. The isomorphism $B_l \otimes B_k \simeq B_k \otimes B_l$ and the energy function are described explicitly in terms of a C_n-analogue of the Robinson-Schensted-Knuth type insertion algorithm. As an application a $C_n^{(1)}$-analogue of the Kostka polynomials is calculated for several cases.

1 Introduction

1.1 Background

Physical combinatorics might be defined naïvely as combinatorics guided by ideas or insights from physics. A distinguished example can be given by the Kostka polynomial. It is a polynomial $K_{\lambda\mu}(q)$ in q depending on two partitions λ, μ with the same number of nodes. Although there are several aspects in this polynomial, one can regard it as a q-analogue of the multiplicity of the irreducible $\mathfrak{sl}_n$-module V_λ in the m-fold tensor product $V_{(\mu_1)} \otimes V_{(\mu_2)} \otimes \cdots \otimes V_{(\mu_m)}$ $(\mu = (\mu_1, \cdots, \mu_m))$. Here for $\lambda = (\lambda_1, \cdots, \lambda_n)$ $(\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$ V_λ denotes the irreducible $\mathfrak{sl}_n$-module with highest weight $\sum_{i=1}^{n-1}(\lambda_i - \lambda_{i+1})\Lambda_i$, Λ_i

*Institute of Physics, University of Tokyo, Komaba, Tokyo 153-8902, Japan

†Department of Informatics and Mathematical Science, Graduate School of Engineering Science, Osaka University, Toyonaka, Osaka 560-8531, Japan

‡Department of Mathematics and Physics, National Defense Academy, Yokosuka 239-8686, Japan

being the fundamental weight of $\mathfrak{sl}_n$. In particular, $V_{(\mu_i)}$ is the symmetric tensor representation of degree μ_i.

In [KR] Kirillov and Reshetikhin presented the following expression for the Kostka polynomial[1]:

$$
\begin{aligned}
K_{\lambda\mu}(q) &= \sum_{\{m\}} q^{c(\{m\})} \prod_{\substack{1\le a\le n-1\\ i\ge 1}} \begin{bmatrix} p_i^{(a)} + m_i^{(a)} \\ m_i^{(a)} \end{bmatrix}, \qquad (1.1)\\
c(\{m\}) &= \frac{1}{2} \sum_{1\le a,b\le n-1} C_{ab} \sum_{i,j\ge 1} \min(i,j) m_i^{(a)} m_j^{(b)} \\
&\qquad - \sum_{i,j\ge 1} \min(i,\mu_j) m_i^{(1)}, \\
p_i^{(a)} &= \delta_{a1} \sum_{j\ge 1} \min(i,\mu_j) - \sum_{1\le b\le n-1} C_{ab} \sum_{j\ge 1} \min(i,j) m_j^{(b)},
\end{aligned}
$$

where the sum $\sum_{\{m\}}$ is taken over $\{m_i^{(a)} \in \mathbb{Z}_{\ge 0} \mid 1 \le a \le n-1, i \ge 1\}$ satisfying $p_i^{(a)} \ge 0$ for $1 \le a \le n-1, i \ge 1$, and $\sum_{i\ge 1} i m_i^{(a)} = \lambda_{a+1} + \lambda_{a+2} + \cdots + \lambda_n$ for $1 \le a \le n-1$. $(C_{ab})_{1\le a,b\le n-1}$ is the Cartan matrix of $\mathfrak{sl}_n$, and $\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right]$ is the q-binomial coefficient or Gaussian polynomial. An intriguing point is that this expression was obtained through the string hypothesis of the Bethe ansatz [Be] for the $\mathfrak{sl}_n$-invariant Heisenberg chain, which is certainly in the field of physics.

Another important idea comes from Baxter's corner transfer matrix (CTM) [Ba, ABF]. In the course of the study of CTM eigenvalues, the notion of one dimensional sum (1dsum) has appeared [DJKMO], and it was recognized that 1dsums give affine Lie algebra characters. Such phenomena were clarified by the theory of perfect crystals [KMN1, KMN2]. As far as the Kostka polynomial is concerned, Nakayashiki and Yamada obtained the following expression [NY]:

$$K_{\lambda\mu}(q) = \sum_{p} q^{E(p)}, \qquad (1.2)$$

where p ranges over the elements $p = b_1 \otimes \cdots \otimes b_m$ of $B_{(\mu_1)} \otimes \cdots \otimes B_{(\mu_m)}$ satisfying $\tilde{e}_i p = 0$ $(i = 1, \cdots, n-1)$ and $wt\, p = \sum_{i=1}^{n-1} (\lambda_i - \lambda_{i+1}) \Lambda_i$. $B_{(\mu_i)}$ is the crystal base of the irreducible $U_q(\mathfrak{sl}_n)$-module with highest weight corresponding to (μ_i) and $\tilde{e}_i$ is the so-called Kashiwara operator. $E(p)$ is called

[1] This expression differs from the conventional definition of $K_{\lambda\mu}(q)$ by an overall power of q.

the energy of p and calculated by using the energy function H as

$$E(p) = \sum_{1\le i<j\le m} H(b_i \otimes b_j^{(i+1)}),$$

where $b_j^{(i)}$ is defined through the crystal isomorphism:

$$\begin{array}{rcl} B_{(\mu_i)} \otimes B_{(\mu_{i+1})} \otimes \cdots \otimes B_{(\mu_j)} & \simeq & B_{(\mu_j)} \otimes B_{(\mu_i)} \otimes \cdots \otimes B_{(\mu_{j-1})} \\ b_i \otimes b_{i+1} \otimes \cdots \otimes b_j & \mapsto & b_j^{(i)} \otimes b'_i \otimes \cdots \otimes b'_{j-1}. \end{array}$$

In the two-fold tensor case, the crystal isomorphism $B_{(\mu_i)} \otimes B_{(\mu_j)} \simeq B_{(\mu_j)} \otimes B_{(\mu_i)} : b_i \otimes b_j \mapsto b'_j \otimes b'_i$ combined with the value $H(b_i \otimes b_j)$ is called the combinatorial R matrix. The crystal base $B_{(l)}$ has a generalization to the rectangular shape $B_{(l^k)}$, and the corresponding generalization of the Kostka polynomial is considered in [SW, S].

In view of the equality (1.1)=(1.2), one is led to an application of the perfect crystal theory of $B_{(l)}$. Define a branching function $b^V_\lambda(q)$ for an $\widehat{\mathfrak{sl}}_n$-module V by

$$\begin{array}{rcl} b^V_\lambda(q) & = & \mathrm{tr}_{\mathcal{H}(V,\lambda)}\, q^{-d}, \\ \mathcal{H}(V,\lambda) & = & \{v \in V \mid e_i v = 0\,(i = 1,\cdots,n-1), \mathrm{wt}\, v = \lambda\}. \end{array}$$

Here d is the degree operator. Let $V(l\Lambda_0)$ be the integrable $\widehat{\mathfrak{sl}}_n$-module with affine highest weight $l\Lambda_0$. Then (1.1)=(1.2) implies the spinon character formula:

$$b^{V(l\Lambda_0)}_\lambda(q) = \sum_\eta \frac{K_{\xi\eta}(q) F^{(l)'}_\eta(q)}{(q)_{\zeta_1}\cdots(q)_{\zeta_{n-1}}}. \tag{1.3}$$

For the definitions of $\xi, (\zeta_1,\cdots,\zeta_{n-1}), F^{(l)'}_\eta(q)$ along with the summing range of η, see Proposition 4.12 of [HKKOTY].

A key to the derivation of (1.3) is the fact that a suitable subset of the semi-infinite tensor product $\cdots \otimes B_{(l)} \otimes \cdots \otimes B_{(l)}$ can be identified with the crystal base $B(l\Lambda_0)$ of the integrable $U_q(\widehat{\mathfrak{sl}}_n)$-module with highest weight $l\Lambda_0$. Since all components are the same, such a case is called homogeneous. Recently, a generalization of such results to inhomogeneous cases is obtained [HKKOT]. For example, a suitable subset of

$$\cdots \otimes B_{(l_1+l_2)} \otimes B_{(l_2)} \otimes \cdots \otimes B_{(l_1+l_2)} \otimes B_{(l_2)} \otimes B_{(l_1+l_2)} \otimes B_{(l_2)}$$

can be identified with $B(l_1\Lambda_0) \otimes B(l_2\Lambda_0)$. Taking the corresponding limit of μ in the equality (1.1)=(1.2), one obtains an expression for the branching function $b^{V(l_1\Lambda_0)\otimes V(l_2\Lambda_0)}_\lambda(q)$.

Another important application of the inhomogeneous case is found in soliton cellular automata. Recently several such automata have been related to known soliton equations through a limiting procedure called ultradiscretization [TS, TTMS]. Although they seem to have nothing to do with the theory of crystals at first view, recent studies revealed their underlying crystal structure [HKT, FOY, HHIKTT]. Namely, the combinatorial R matrix appears as the scattering rule of solitons as well as the time evolution operator for the automaton.

1.2 Present work

In the $\widehat{\mathfrak{sl}}_n$ case, a typical example of the isomorphism $B_{(3)} \otimes B_{(2)} \simeq B_{(2)} \otimes B_{(3)}$ is

$$112 \otimes 23 \mapsto 12 \otimes 123.$$

It may be viewed as a scattering process of two composite particles 112 and 23. Through the collision the constituent particles are re-shuffled and then recombined into two other composite particles 12 and 123.

In this paper we study the combinatorial R matrices for a family of $U'_q(C_n^{(1)})$ and $U'_q(A_{2n-1}^{(2)})$ crystals. This includes a new type of examples as

$$\begin{aligned} 123 \otimes \bar{2}\bar{1} &\mapsto 23 \otimes 0\bar{2}\bar{0} \quad \text{for } U'_q(C_n^{(1)}) \text{ case,} \\ &\mapsto 13 \otimes 1\bar{1}\bar{1} \quad \text{for } U'_q(A_{2n-1}^{(2)}) \text{ case.} \end{aligned}$$

Here we observe "anti-particles", which undergo a pair annihilation and a pair creation: $(1) + (\bar{1}) \longrightarrow (0) + (\bar{0})$ or $(2) + (\bar{2}) \longrightarrow (1) + (\bar{1})$.

We shall consider a family $\{B_l \mid l \in \mathbb{Z}_{\geq 1}\}$ of crystals for $U'_q(C_n^{(1)})$ and $U'_q(A_{2n-1}^{(2)})$. The above example corresponds to $B_3 \otimes B_2 \simeq B_2 \otimes B_3$. Here B_l is the crystal of the irreducible U'_q-module corresponding to the l-fold symmetric "fusion" of the vector representation. For $U'_q(A_{2n-1}^{(2)})$ it was constructed in [KKM]. For $U'_q(C_n^{(1)})$, B_l in this paper denotes $B_{l/2}$ in [KKM] (B_l in [HKKOT]) when l is even (odd). Our main result is the explicit description of the isomorphism $B_l \otimes B_k \simeq B_k \otimes B_l$ and the associated energy function for any l and k. It will be done through a slight modification of the insertion algorithm for the C-tableaux introduced by T. H. Baker [B]. Since the two affine algebras $C_n^{(1)}$ and $A_{2n-1}^{(2)}$ share the common classical part C_n, they allow a parallel treatment and the results are similar in many respects. Let us sketch them along the content of the paper.

In Section 2, we recall some basic facts about crystals. As a $U_q(C_n)$ crystal, it is known that $U'_q(C_n^{(1)})$ crystal B_l decomposes into the disjoint

union of $B(l\Lambda_1), B((l-2)\Lambda_1), \ldots,$ where $B(\lambda)$ denotes the crystal of the irreducible representation with highest weight λ. Within each $B(l'\Lambda_1)$ it is natural [KN, B] to parametrize the elements by length l' one-row semistandard tableaux with letters $1 < \cdots n < \overline{n} < \cdots \overline{2} < \overline{1}$. Instead of doing so we will represent elements in B_l uniformly via length l one-row semistandard tableaux with letters $0 < 1 < \cdots n < \overline{n} < \cdots \overline{2} < \overline{1} < \overline{0}$. Here the number x_0 of 0 and $\overline{x}_0$ of $\overline{0}$ must be the same, according to which the elements belong to $B((l-2x_0)\Lambda_1)$. Thus the number of letters in the tableaux has increased from $2n$ to $2(n+1)$. In fact, under the insertion scheme in later sections, these tableaux will behave like those for $U_q(C_{n+1})$ [B] in some sense.

In Section 3 we first define an insertion algorithm for the tableaux introduced in Section 2. When there is no $(x, \overline{x})$ pair, it is the same as the well known $\mathfrak{sl}_n$ case [F]. In general, our algorithm is essentially Baker's one [B] for $U_q(C_{n+1})$ if $0 < \cdots < n < \overline{n} < \cdots < \overline{0}$ is regarded as $1 < \cdots < n+1 < \overline{n+1} < \cdots < \overline{1}$. See Remark 3.2. We describe it only for those tableaux with depth at most two, which suffices for our aim. We then state a main theorem, which describes the combinatorial R matrix of $U'_q(C_n^{(1)})$ explicitly in terms of the insertion scheme.

In Section 4 we prove the main theorem. As a $U_q(C_n)$ crystal, $B_l \otimes B_k$ decomposes into connected components which are isomorphic to the crystals of irreducible $U_q(C_n)$-modules. Within each component the general elements are obtained by applying $\tilde{f}_i$'s $(1 \leq i \leq n)$ to the $U_q(C_n)$ highest elements. Our strategy is first to verify the theorem directly for the highest elements. For general elements the theorem follows from the fact due to Baker that our insertion algorithm on letters $0, 1, \ldots, \overline{1}, \overline{0}$ can be regarded as the isomorphism of $U_q(C_{n+1})$ crystals. It turns out that $B_l \supset B_{l-2} \supset B_{l-4} \supset \cdots$ as the sets according to the number of $(0, \overline{0})$ pairs contained in the tableaux. We shall utilize this fact to remove the $(0, \overline{0})$ pairs before the insertion so as to avoid the pair annihilation of the boxes under the insertions and the resulting bumping-sliding transition in [B].

In Section 5, a parallel treatment is done for $U'_q(A^{(2)}_{2n-1})$. This case is simpler in that B_l coincides with $B(l\Lambda_1)$ as a $U_q(C_n)$ crystal. Consequently we do not have letters 0 and $\overline{0}$ in the tableaux. The main difference from $U'_q(C^{(1)})$ case is to remove 1 and $\overline{1}$ appropriately before the insertion.

In Appendix A, we detail the calculation for the proof of Proposition 4.1.

In Appendix B, another rule for finding the image under $B_l \otimes B_k \simeq B_k \otimes B_l$ is given for $U'_q(C_n)$ case. In practical calculations it is often more efficient than the one based on the insertion scheme in the main text.

In Appendix C, the $C_n^{(1)}$-analogue $X_{\lambda,\mu}(t)$ of the Kostka polynomials in the sense of Section 1.1 is listed up to $|\mu| = 6$. They coincide with the Kostka

polynomial if $|\lambda| = |\mu|$.

We remark that the isomorphism $B_l \otimes B_k \simeq B_k \otimes B_l$ for $U'_q(C_n^{(1)})$ in this paper has been identified with the two body scattering rule in the soliton cellular automaton [HKT].

Acknowledgements The authors thank T. H. Baker for useful discussions. They also thank M. Kashiwara and T. Miwa for organizing the conference "Physical Combinatorics" in Kyoto during January 28 - February 2, 1999.

2 Definitions

2.1 Brief summary of crystals

Let I be an index set. A crystal B is a set B with the maps

$$\tilde{e}_i, \tilde{f}_i : B \sqcup \{0\} \longrightarrow B \sqcup \{0\} \quad (i \in I)$$

satisfying the following properties:

$\tilde{e}_i 0 = \tilde{f}_i 0 = 0$,

for any b and i, there exists $n > 0$ such that $\tilde{e}_i^n b = \tilde{f}_i^n b = 0$,

for $b, b' \in B$ and $i \in I$, $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$.

For an element b of B we set

$$\varepsilon_i(b) = \max\{n \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_i^n b \neq 0\}, \quad \varphi_i(b) = \max\{n \in \mathbb{Z}_{\geq 0} \mid \tilde{f}_i^n b \neq 0\}.$$

For two crystals B and B', the tensor product $B \otimes B'$ is defined.

$$B \otimes B' = \{b \otimes b' \mid b \in B, b' \in B'\}.$$

The actions of $\tilde{e}_i$ and $\tilde{f}_i$ are defined by

$$\tilde{e}_i(b \otimes b') = \begin{cases} \tilde{e}_i b \otimes b' & \text{if } \varphi_i(b) \geq \varepsilon_i(b') \\ b \otimes \tilde{e}_i b' & \text{if } \varphi_i(b) < \varepsilon_i(b'), \end{cases} \tag{2.1}$$

$$\tilde{f}_i(b \otimes b') = \begin{cases} \tilde{f}_i b \otimes b' & \text{if } \varphi_i(b) > \varepsilon_i(b') \\ b \otimes \tilde{f}_i b' & \text{if } \varphi_i(b) \leq \varepsilon_i(b'). \end{cases} \tag{2.2}$$

Here $0 \otimes b$ and $b \otimes 0$ are understood to be 0.

2.2 Energy function and combinatorial R matrix

Let $\mathfrak{g}$ be an affine Lie algebra and let B and B' be two $U_q'(\mathfrak{g})$ crystals. We assume that B and B' are finite sets, and that $B \otimes B'$ is connected. The algebra $U_q'(\mathfrak{g})$ is a subalgebra of $U_q(\mathfrak{g})$. Their definitions are given in Section 2.1 (resp. 3.2) of [KMN1] for $U_q(\mathfrak{g})$ (resp. $U_q'(\mathfrak{g})$).

Suppose $b \otimes b' \in B \otimes B'$ is mapped to $\tilde{b}' \otimes \tilde{b} \in B' \otimes B$ under the isomorphism $B \otimes B' \simeq B' \otimes B$ of $U_q'(\mathfrak{g})$ crystals. A $\mathbb{Z}$-valued function H on $B \otimes B'$ is called an *energy function* if for any i and $b \otimes b' \in B \otimes B'$ such that $\tilde{e}_i(b \otimes b') \neq 0$, it satisfies

$$H(\tilde{e}_i(b \otimes b')) = \begin{cases} H(b \otimes b') + 1 & \text{if } i = 0, \varphi_0(b) \geq \varepsilon_0(b'), \varphi_0(\tilde{b}') \geq \varepsilon_0(\tilde{b}), \\ H(b \otimes b') - 1 & \text{if } i = 0, \varphi_0(b) < \varepsilon_0(b'), \varphi_0(\tilde{b}') < \varepsilon_0(\tilde{b}), \\ H(b \otimes b') & \text{otherwise.} \end{cases} \tag{2.3}$$

When we want to emphasize $B \otimes B'$, we write $H_{BB'}$ for H. This definition of the energy function is due to (3.4.e) of [NY], which is a generalization of the definition for $B = B'$ case in [KMN1]. The energy function is unique up to an additive constant, since $B \otimes B'$ is connected. By definition, $H_{BB'}(b \otimes b') - H_{B'B}(\tilde{b}' \otimes \tilde{b})$ is a constant independent of $b \otimes b'$. In this paper we choose the constant to be 0. We call the isomorphism $B \otimes B' \simeq B' \otimes B$ endowed with the energy function $H_{BB'}$ the *combinatorial R-matrix*.

2.3 $C_n^{(1)}$ crystals

Given a non-negative integer l, we consider a $U_q'(C_n^{(1)})$ crystal denoted by B_l. If l is even, B_l is the same as that defined in [KKM]. (Their B_l is identical to our B_{2l}.) If l is odd, B_l is defined in [HKKOT]. B_l's are the crystals associated with the crystal bases of the irreducible finite dimensional representation of the quantum affine algebra $U_q'(C_n^{(1)})$. As a set B_l reads

$$B_l = \left\{ (x_1, \ldots, x_n, \overline{x}_n, \ldots, \overline{x}_1) \;\middle|\; x_i, \overline{x}_i \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^{n} (x_i + \overline{x}_i) \in \{l, l-2, \ldots\} \right\}.$$

The crystal structure is given by (2.7).

B_l is isomorphic to $\bigoplus_{0 \leq j \leq l,\, j \equiv l \pmod 2} B(j\Lambda_1)$ as crystals for $U_q(C_n)$, where $B(j\Lambda_1)$ is the one associated with the irreducible representation of with highest weight $j\Lambda_1$. As a special case of the more general family of $U_q(C_n)$ crystals [KN], the crystal $B(j\Lambda_1)$ has a description with the semistandard C-tableaux. The entries are $1, \ldots, n$ and $\overline{1}, \ldots, \overline{n}$, with the total order:

$$1 < 2 < \cdots < n < \overline{n} < \cdots < \overline{2} < \overline{1}.$$

In this description $b=(x_1,\dots,x_n,\overline{x}_n,\dots,\overline{x}_1)\in B(j\Lambda_1)$ is depicted by

$$b=\overbrace{\boxed{1\cdots1}}^{x_1}\boxed{\cdots}\overbrace{\boxed{n\cdots n}}^{x_n}\overbrace{\boxed{\overline{n}\cdots\overline{n}}}^{\overline{x}_n}\boxed{\cdots}\overbrace{\boxed{\overline{1}\cdots\overline{1}}}^{\overline{x}_1}. \tag{2.4}$$

The length of this one-row tableau is equal to j, namely $\sum_{i=1}^{n}(x_i+\overline{x}_i)=j$.

Here and in the remaining part of this paper we denote $\overbrace{\boxed{i}\boxed{i}\boxed{\cdots}\boxed{i}}^{x}$ by

$$\overbrace{\boxed{i\cdots i}}^{x} \text{ or more simply by } \overset{x}{\boxed{\quad i\quad}}\ .$$

We shall depict the elements of B_l by one-row tableaux with length l, by supplying pairs of $\boxed{0}$ and $\boxed{\overline{0}}$. Adding 0 and $\overline{0}$ into the set of the entries of the tableaux, we assume the total order $0<1<\cdots<\overline{1}<\overline{0}$. Thus we depict $b=(x_1,\dots,x_n,\overline{x}_n,\dots,\overline{x}_1)\in B_l$ by

$$\mathbb{T}(b)=\overbrace{\boxed{0\cdots0}}^{x_0}\overbrace{\boxed{1\cdots1}}^{x_1}\boxed{\cdots}\overbrace{\boxed{n\cdots n}}^{x_n}\overbrace{\boxed{\overline{n}\cdots\overline{n}}}^{\overline{x}_n}\boxed{\cdots}\overbrace{\boxed{\overline{1}\cdots\overline{1}}}^{\overline{x}_1}\overbrace{\boxed{\overline{0}\cdots\overline{0}}}^{\overline{x}_0}, \tag{2.5}$$

where $x_0=\overline{x}_0=(l-\sum_{i=1}^{n}(x_i+\overline{x}_i))/2$. If $x_0=0$ we say that $\mathbb{T}(b)$ has no $\boxed{0}$. Sometimes we identify $\mathbb{T}(b)$ with b, and omit the frame of $\mathbb{T}(b)$, e.g., $0\overline{2}\overline{0}=(0,\dots,0,1,0)\in B_3$.

This description means that we have embedded B_l, as a set, into $U_q(C_{n+1})$ crystal $B(l\Lambda_1)$. Let us denote by ς this embedding:

$$\varsigma: U_q(C_n) \text{ crystal } B_l \text{ as a set} \hookrightarrow U_q(C_{n+1}) \text{ crystal } B(l\Lambda_1). \tag{2.6}$$

It shifts the entries of the tableaux as $\varsigma(i)=i+1$ and $\varsigma(\overline{i})=\overline{i+1}$ for $i=0,1,\dots,n$. For example $\varsigma(0\overline{2}\overline{0})=1\overline{3}\overline{1}$. For $b\in B_l$ and $i=1,\dots,n$ one has $\varsigma(\tilde{e}_ib)=\tilde{e}_{i+1}\varsigma(b)$ and $\varsigma(\tilde{f}_ib)=\tilde{f}_{i+1}\varsigma(b)$.

The crystal structure of B_l is give by

$$\begin{aligned}
\tilde{e}_0b &= \begin{cases}(x_1-2,x_2,\dots,\overline{x}_2,\overline{x}_1) & \text{if } x_1\ge\overline{x}_1+2,\\ (x_1-1,x_2,\dots,\overline{x}_2,\overline{x}_1+1) & \text{if } x_1=\overline{x}_1+1,\\ (x_1,x_2,\dots,\overline{x}_2,\overline{x}_1+2) & \text{if } x_1\le\overline{x}_1,\end{cases}\\
\tilde{e}_nb &= (x_1,\dots,x_n+1,\overline{x}_n-1,\dots,\overline{x}_1),\\
\tilde{e}_ib &= \begin{cases}(x_1,\dots,x_i+1,x_{i+1}-1,\dots,\overline{x}_1) & \text{if } x_{i+1}>\overline{x}_{i+1},\\ (x_1,\dots,\overline{x}_{i+1}+1,\overline{x}_i-1,\dots,\overline{x}_1) & \text{if } x_{i+1}\le\overline{x}_{i+1},\end{cases}
\end{aligned}$$

$$\begin{aligned}
\tilde{f}_0 b &= \begin{cases} (x_1+2, x_2, \ldots, \overline{x}_2, \overline{x}_1) & \text{if } x_1 \geq \overline{x}_1, \\ (x_1+1, x_2, \ldots, \overline{x}_2, \overline{x}_1 - 1) & \text{if } x_1 = \overline{x}_1 - 1, \\ (x_1, x_2, \ldots, \overline{x}_2, \overline{x}_1 - 2) & \text{if } x_1 \leq \overline{x}_1 - 2, \end{cases} \\
\tilde{f}_n b &= (x_1, \ldots, x_n - 1, \overline{x}_n + 1, \ldots, \overline{x}_1), \\
\tilde{f}_i b &= \begin{cases} (x_1, \ldots, x_i - 1, x_{i+1} + 1, \ldots, \overline{x}_1) & \text{if } x_{i+1} \geq \overline{x}_{i+1}, \\ (x_1, \ldots, \overline{x}_{i+1} - 1, \overline{x}_i + 1, \ldots, \overline{x}_1) & \text{if } x_{i+1} < \overline{x}_{i+1}, \end{cases}
\end{aligned} \tag{2.7}$$

where $b = (x_1, \ldots, x_n, \overline{x}_n, \ldots, \overline{x}_1)$ and $i = 1, \ldots, n-1$. For this b we have

$$\begin{aligned}
\varphi_i(b) &= x_i + (\overline{x}_{i+1} - x_{i+1})_+ \quad \text{for } i = 0, 1, \ldots, n-1, \\
\varepsilon_i(b) &= \overline{x}_i + (x_{i+1} - \overline{x}_{i+1})_+ \quad \text{for } i = 0, 1, \ldots, n-1, \\
\varphi_n(b) &= x_n, \quad \varepsilon_n(b) = \overline{x}_n.
\end{aligned} \tag{2.8}$$

Here $(x)_+ := \max(x, 0)$.

Except for Section 5 concerning $A^{(2)}_{2n-1}$, we normalize the energy function for $C^{(1)}_n$ case as

$$H_{B_l B_k}((l, 0, \ldots, 0) \otimes (0, k, 0, \ldots, 0)) = 0,$$

irrespective of $l < k$ or $l \geq k$. (For $A^{(2)}_{2n-1}$ we will employ a different normalization. See (5.4).)

3 Explicit description of isomorphism and energy function

3.1 The algorithm of column insertions

Set an alphabet $\mathcal{X} = \mathcal{A} \sqcup \bar{\mathcal{A}}$, $\mathcal{A} = \{0, 1, \ldots, n\}$ and $\bar{\mathcal{A}} = \{\overline{0}, \overline{1}, \ldots, \overline{n}\}$, with the total order $0 < 1 < \cdots < n < \overline{n} < \cdots < \overline{1} < \overline{0}$. Unless otherwise stated, a tableau means a (column-strict) semistandard one with entries taken from $\mathcal{X}$ in Section 3 and 4. For the alphabet $\mathcal{X}$, we follow the convention that Greek letters $\alpha, \beta, \ldots$ belong to $\mathcal{A} \sqcup \bar{\mathcal{A}}$ while Latin letters $x, y, \ldots$ (resp. $\overline{x}, \overline{y}, \ldots$) belong to $\mathcal{A}$ (resp. $\bar{\mathcal{A}}$).

Given a letter $\alpha \in \mathcal{X}$ and the tableau T that have at most two rows, we define a tableau denoted by $(\boxed{\alpha} \to T)$, and call such an algorithm a "column insertion of a letter α into a tableau T ". (We sometimes identify a letter α with a box $\boxed{\alpha}$.) Let us begin with such T's that have at most one column. The procedure of the column insertion $(\boxed{\alpha} \to T)$ can be summarized as follows:

$$\text{case 1a}\ \left(\boxed{\alpha}\to\ \emptyset\ \right)=\boxed{\alpha}\qquad ,$$

$$\text{case 2a}\ \left(\boxed{\beta}\to\boxed{\alpha}\right)=\begin{array}{|c|}\hline \alpha\\ \hline \beta\\ \hline\end{array}\qquad \text{if } \alpha<\beta,$$

$$\text{case 1b}\ \left(\boxed{\alpha}\to\boxed{\beta}\right)=\begin{array}{|c|c|}\hline \alpha & \beta\\ \hline\end{array}\qquad \text{if } \alpha\le\beta,$$

$$\text{case 2b}\ \left(\boxed{\beta}\to\begin{array}{|c|}\hline \alpha\\ \hline \gamma\\ \hline\end{array}\right)=\begin{array}{|c|c|}\hline \alpha & \gamma\\ \hline \beta & \multicolumn{1}{|c}{}\\ \cline{1-1}\end{array}\qquad \text{if } \alpha<\beta\le\gamma \text{ and } (\alpha,\gamma)\neq(x,\overline{x}),$$

$$\text{case 3b}\ \left(\boxed{\alpha}\to\begin{array}{|c|}\hline \beta\\ \hline \gamma\\ \hline\end{array}\right)=\begin{array}{|c|c|}\hline \alpha & \beta\\ \hline \gamma & \multicolumn{1}{|c}{}\\ \cline{1-1}\end{array}\qquad \text{if } \alpha\le\beta<\gamma \text{ and } (\alpha,\gamma)\neq(x,\overline{x}),$$

$$\text{case 4b}\ \left(\boxed{\beta}\to\begin{array}{|c|}\hline x\\ \hline \overline{x}\\ \hline\end{array}\right)=\begin{array}{|c|c|}\hline x-1 & \overline{x-1}\\ \hline \beta & \multicolumn{1}{|c}{}\\ \cline{1-1}\end{array}\qquad \text{if } x\le\beta\le\overline{x} \text{ and } x\neq 0,$$

$$\text{case 5b}\ \left(\boxed{x}\to\begin{array}{|c|}\hline \beta\\ \hline \overline{x}\\ \hline\end{array}\right)=\begin{array}{|c|c|}\hline x+1 & \beta\\ \hline \overline{x+1} & \multicolumn{1}{|c}{}\\ \cline{1-1}\end{array}\qquad \text{if } x<\beta<\overline{x} \text{ and } x\neq n.$$

The cases 2b - 5b do not cover all the tableaux with two rows, but we only deal with these situations in this paper. In particular, the tableaux generated by these insertions have at most two rows. Note that the algorithm except for the cases 4b and 5b agrees with the Knuth-type column insertion. We call the cases 1b - 5b the "bumping cases".

When T is a general tableau with at most two rows, we repeat the above procedure: we insert a box into the leftmost column of T according to the above formula. If it is not a bumping case, replace the column by the right hand side of the formula. Otherwise, replace the column by the right hand side of the formula without the right box. We regard that this right box is bumped. We insert it into the second column of T from the left and repeat the procedure above until we come to a non-bumping case 1a or 2a.

Example 3.1. $n=4$.

$$\left(\boxed{2}\to\begin{array}{|c|c|c|c|}\hline 0&3&4&\bar{0}\\ \hline 4&\bar{3}&\bar{1}&\multicolumn{1}{|c}{}\\ \cline{1-3}\end{array}\right)=\underset{\substack{\uparrow\ \ \ \ \ \ \ \\ \boxed{4}\ \ \ \ \ \ \ }}{\begin{array}{|c|c|c|c|}\hline 0&3&4&\bar{0}\\ \hline 2&\bar{3}&\bar{1}&\multicolumn{1}{|c}{}\\ \cline{1-3}\end{array}}=\underset{\substack{\ \ \ \ \uparrow\\ \ \ \ \ \boxed{\bar{2}}}}{\begin{array}{|c|c|c|c|}\hline 0&2&4&\bar{0}\\ \hline 2&4&\bar{1}&\multicolumn{1}{|c}{}\\ \cline{1-3}\end{array}}=\begin{array}{|c|c|c|c|}\hline 0&2&4&\bar{0}\\ \hline 2&4&\bar{2}&\multicolumn{1}{|c}{}\\ \cline{1-3}\end{array}\underset{\substack{\uparrow\\ \boxed{\bar{1}}}}{}=\begin{array}{|c|c|c|c|c|}\hline 0&2&4&\bar{1}&\bar{0}\\ \hline 2&4&\bar{2}&\multicolumn{2}{|c}{}\\ \cline{1-3}\end{array}$$

For a tableau T we denote by $w(T)$ the Japanese reading word of T. The $w(T)$ is a sequence of letters that is created by reading all letters on T from the rightmost column to the leftmost column, and in each column from the top to the bottom. For instance,

$$w(\begin{array}{|c|c|c|c|}\hline \alpha_1&\alpha_2&\cdots&\alpha_j\\\hline\end{array}) = \alpha_j\cdots\alpha_2\alpha_1,$$

and

$$w\left(\begin{array}{|c|c|c|c|c|c|c|}\hline \alpha_1&\alpha_2&\cdots&\alpha_i&\alpha_{i+1}&\cdots&\alpha_j\\\hline \beta_1&\beta_2&\cdots&\beta_i\\\cline{1-4}\end{array}\right) = \alpha_j\cdots\alpha_{i+1}\alpha_i\beta_i\cdots\alpha_2\beta_2\alpha_1\beta_1,$$

and so on. Let T and T' be one-row tableaux. By abuse of notation we denote by $T' \to T$ the tableau constructed by successive column insertions of the letters of the word $w(T')$ into T. Namely if

$$w(T') = \tau_j\tau_{j-1}\cdots\tau_1,$$

then we write

$$(T' \to T) = (\tau_1 \to \cdots(\tau_{j-1} \to (\tau_j \to T))\cdots).$$

(Following a usual convention in type A [F], it might be written as a *product tableau*, $T' \cdot T$.) In particular $(T \to \emptyset) = T$ for any T.

Throughout this paper we let T_1 be the length of the first row of a tableau T.

Remark 3.2. Our algorithm is a specialization of the column insertion for C_n-case [B]. Let b_i be an element of $U_q(C_{n+1})$ crystal $B(l_i\Lambda_1)\,(i = 1,2)$. Denote by $b_2 \xrightarrow{*} b_1$ the tableau obtained by successive column insertions with the original definition [B] of $w(\mathbb{T}(b_2))$ into b_1. (In [B], $b_2 \xrightarrow{*} b_1$ is denoted by $b_1 * b_2$.) Then, for any element b_i of $U'_q(C^{(1)}_n)$ crystal B_{l_i} such that $\mathbb{T}(b_1)$ or $\mathbb{T}(b_2)$ has no $\boxed{0}$, our $\big(\mathbb{T}(b_2) \to \mathbb{T}(b_1)\big)$ has been determined so that

$$\varsigma\Big(\big(\mathbb{T}(b_2) \to \mathbb{T}(b_1)\big)\Big) = \varsigma\big(\mathbb{T}(b_2)\big) \xrightarrow{*} \varsigma\big(\mathbb{T}(b_1)\big).$$

We will calculate $\big(\mathbb{T}(b_2) \to \mathbb{T}(b_1)\big)$ only when $\mathbb{T}(b_1)$ or $\mathbb{T}(b_2)$ has no $\boxed{0}$ in this paper. Under such a situation no pair annihilation takes place during the insertion in the right hand side. See also Remark 4.9.

We shall also use the reverse bumping algorithm [B]. In our case where the tableau has at most two rows, the algorithm is rather simple. We only use them in the following five cases.

$$\text{case 1c}\quad \begin{array}{c}\boxed{\beta}\\ \downarrow\\ \boxed{\alpha}\end{array} \quad = \quad \begin{array}{cc}\boxed{\alpha}&\\ \downarrow&\\ &\boxed{\beta}\end{array} \quad \text{if } \alpha \le \beta,$$

$$\text{case 2c}\quad \begin{array}{c}\boxed{\gamma}\\ \downarrow \\ \begin{array}{|c|}\hline \alpha\\ \hline \beta\\ \hline\end{array}\end{array} \;=\; \begin{array}{c}\boxed{\beta}\\ \downarrow\\ \begin{array}{|c|}\hline \alpha \\ \hline \gamma\\ \hline\end{array}\end{array} \quad \text{if } \alpha<\beta\le\gamma \text{ and } (\alpha,\gamma)\neq(x,\overline{x}),$$

$$\text{case 3c}\quad \begin{array}{c}\boxed{\beta}\\ \downarrow \\ \begin{array}{|c|}\hline \alpha\\ \hline \gamma\\ \hline\end{array}\end{array} \;=\; \begin{array}{c}\boxed{\alpha}\\ \downarrow\\ \begin{array}{|c|}\hline \beta \\ \hline \gamma\\ \hline\end{array}\end{array} \quad \text{if } \alpha\le\beta<\gamma \text{ and } (\alpha,\gamma)\neq(x,\overline{x}),$$

$$\text{case 4c}\quad \begin{array}{c}\boxed{\overline{x}}\\ \downarrow \\ \begin{array}{|c|}\hline x\\ \hline \beta\\ \hline\end{array}\end{array} \;=\; \begin{array}{c}\boxed{\beta}\\ \downarrow\\ \begin{array}{|c|}\hline x+1 \\ \hline \overline{x+1}\\ \hline\end{array}\end{array} \quad \text{if } x<\beta<\overline{x} \text{ and } x\neq n,$$

$$\text{case 5c}\quad \begin{array}{c}\boxed{\beta}\\ \downarrow \\ \begin{array}{|c|}\hline x\\ \hline \overline{x}\\ \hline\end{array}\end{array} \;=\; \begin{array}{c}\boxed{x-1}\\ \downarrow\\ \begin{array}{|c|}\hline \beta \\ \hline \overline{x-1}\\ \hline\end{array}\end{array} \quad \text{if } x\le\beta\le\overline{x} \text{ and } x\neq 0,$$

where $\begin{array}{c}\boxed{\alpha}\\ \downarrow\\ C\end{array} = \begin{array}{c}\boxed{\beta}\\ \downarrow\\ C'\end{array}$ means that if a letter β is column inserted into a column C' then the column is changed to C and a letter α is bumped out.

3.2 Main theorem : $C_n^{(1)}$ case

Fix $l, k \in \mathbb{Z}_{\geq 1}$. Given $b_1 \otimes b_2 \in B_l \otimes B_k$, we define the element $b_2' \otimes b_1' \in B_k \otimes B_l$ and $l', k', m \in \mathbb{Z}_{\geq 0}$ by the following rule.

Rule 3.3.
Set $z = \min(\sharp\boxed{0} \text{ in } \mathbb{T}(b_1), \sharp\boxed{0} \text{ in } \mathbb{T}(b_2)) = \min(\sharp\boxed{\overline{0}} \text{ in } \mathbb{T}(b_1), \sharp\boxed{\overline{0}} \text{ in } \mathbb{T}(b_2))$. Remove $(\boxed{0}, \boxed{\overline{0}})$ pairs simultaneously from $\mathbb{T}(b_1)$ and $\mathbb{T}(b_2)$ z times. Denote the resulting tableaux by $\hat{\mathbb{T}}(b_1)$ and $\hat{\mathbb{T}}(b_2)$, and set $l' = \hat{\mathbb{T}}(b_1)_1 = l - 2z$ and $k' = \hat{\mathbb{T}}(b_2)_1 = k - 2z$. ($T_1$ is the length of the first row of a tableau T.)

Operate the column insertion and set $\hat{\mathbb{P}}(b_2 \to b_1) = \Big(\hat{\mathbb{T}}(b_2) \longrightarrow \hat{\mathbb{T}}(b_1)\Big)$. It has the form:

$$\begin{array}{|c|c|}\hline j_1 \cdots\cdots j_{k'} & i_{m+1} \;\cdots\; i_{l'} \\ \hline i_1 \cdots i_m & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

where m is the length of the second row, hence that of the first row is $l' + k' - m$. $(0 \le m \le k'.)$

Next we bump out l' letters from the tableau $T^{(0)} = \hat{\mathbb{P}}(b_2 \to b_1)$ by the reverse bumping algorithm. For the boxes containing $i_{l'}, i_{l'-1}, \ldots, i_1$ in the above tableau, we do it first for $i_{l'}$ then $i_{l'-1}$ and so on. Correspondingly, let w_1 be the first letter that is bumped out from the leftmost column and w_2 be the second and so on. Denote by $T^{(i)}$ the resulting tableau when w_i is bumped out $(1 \le i \le l')$. Note that $w_1 \le w_2 \le \cdots \le w_{l'}$. Now $b'_1 \in B_l$ and $b'_2 \in B_k$ are uniquely specified by

$$\begin{aligned}
\mathbb{T}(b'_2) &= \begin{array}{|c|c|c|}\multicolumn{1}{c}{z} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{z} \\ \hline 0\cdots 0 & T^{(l')} & \bar{0}\cdots\bar{0} \\ \hline \end{array}\ , \\
\mathbb{T}(b'_1) &= \begin{array}{|c|c|c|c|c|}\multicolumn{1}{c}{z} & \multicolumn{3}{c}{} & \multicolumn{1}{c}{z} \\ \hline 0\cdots 0 & w_1 & \cdots & w_{l'} & \bar{0}\cdots\bar{0} \\ \hline \end{array}\ .
\end{aligned}$$

Our main result for $U'_q(C_n^{(1)})$ is

Theorem 3.4. *Given $b_1 \otimes b_2 \in B_l \otimes B_k$, specify $b'_2 \otimes b'_1 \in B_k \otimes B_l$ and l', k', m by Rule 3.3. Let $\iota : B_l \otimes B_k \xrightarrow{\sim} B_k \otimes B_l$ be the isomorphism of $U'_q(C_n^{(1)})$ crystal. Then we have*

$$\begin{aligned}
\iota(b_1 \otimes b_2) &= b'_2 \otimes b'_1, \\
H_{B_l B_k}(b_1 \otimes b_2) &= \min(l', k') - m.
\end{aligned}$$

In Appendix B we give an alternative algorithm equivalent to Rule 3.3, which is analogous to the type A case (Rule 3.11 of [NY]). In practical calculations it is often more efficient than Rule 3.3 based on the insertion algorithm.

Remark 3.5. Associated with the tableau $\hat{\mathbb{P}}(b_2 \to b_1) = \Big(\hat{\mathbb{T}}(b_2) \longrightarrow \hat{\mathbb{T}}(b_1)\Big)$, we have the recording tableau $\hat{\mathbb{Q}}(b_2 \to b_1)$, as in the Robinson-Schensted-Knuth correspondence [F]. $\hat{\mathbb{Q}}(b_2 \to b_1)$ has a common shape with $\hat{\mathbb{P}}(b_2 \to b_1)$, and its entries are the consecutive integers from 1 to $l' + k'$. (Integers from $l' + 1$ to $l' + m$ are in the second row.) With $\hat{\mathbb{Q}}(b_2 \to b_1)$, we can reverse

the column insertion procedure and recover $\hat{\mathbb{T}}(b_1)$ and $\hat{\mathbb{T}}(b_2)$ from $\hat{\mathbb{P}}(b_2 \to b_1)$. The recording tableau $\hat{\mathbb{Q}}(b'_1 \to b'_2)$ for the column insertion $\left(\hat{\mathbb{T}}(b'_1) \longrightarrow \hat{\mathbb{T}}(b'_2)\right)$ is similarly defined. It has a common shape with $\hat{\mathbb{P}}(b_2 \to b_1)$, and its entries are the consecutive integers from 1 to $l' + k'$. (Integers from $k'+1$ to $k'+m$ are in the second row.) In Rule 3.3 we have constructed $T^{(l')}$ and $\boxed{w_1 \cdots w_{l'}}$ from $\hat{\mathbb{P}}(b_2 \to b_1)$ with the help of the recording tableau $\hat{\mathbb{Q}}(b'_1 \to b'_2)$.

Example 3.6. Let us assume $n \geq 4$ and take $b_1 = (2,0,1,1,0,\ldots,0,1,1,1) \in B_9$ and $b_2 = (0,\ldots,0,3,0,0,2) \in B_7$. Then we have $z = 1, l' = 7, k' = 5$ and

$$\hat{\mathbb{T}}(b_1) = 1134\bar{3}\bar{2}\bar{1}, \quad \hat{\mathbb{T}}(b_2) = \bar{4}\bar{4}\bar{4}\bar{1}\bar{1}.$$

In this example we have

$$\hat{\mathbb{P}}(b_2 \to b_1) = \begin{array}{l} 0034\bar{3}\bar{2}\bar{1} \\ \bar{4}\bar{4}\bar{4}\bar{0}\bar{0} \end{array}, \ \hat{\mathbb{Q}}(b_2 \to b_1) = \begin{array}{l} 1234567 \\ 89\acute{0}\acute{1}\acute{2} \end{array}, \ \hat{\mathbb{Q}}(b'_1 \to b'_2) = \begin{array}{l} 12345\acute{1}\acute{2} \\ 67890\acute{0} \end{array}.$$

Here we have written $10, 11, 12$ as $\acute{0}, \acute{1}, \acute{2}$. The column insertion $\left(\hat{\mathbb{T}}(b_2) \longrightarrow \hat{\mathbb{T}}(b_1)\right)$ goes as

$$\begin{array}{l} 1134\bar{3}\bar{2}\bar{1} \\ \bar{1} \end{array}, \quad \begin{array}{l} 0134\bar{3}\bar{2}\bar{1} \\ \bar{1}\bar{0} \end{array}, \quad \begin{array}{l} 0134\bar{3}\bar{2}\bar{1} \\ \bar{4}\bar{1}\bar{0} \end{array}, \quad \begin{array}{l} 0034\bar{3}\bar{2}\bar{1} \\ \bar{4}\bar{4}\bar{0}\bar{0} \end{array}, \quad \begin{array}{l} 0034\bar{3}\bar{2}\bar{1} \\ \bar{4}\bar{4}\bar{4}\bar{0}\bar{0} \end{array}.$$

The reverse bumping according to the recording tableau $\hat{\mathbb{Q}}(b'_1 \to b'_2)$ goes as

$$\begin{array}{l} 0034\bar{3}\bar{2}\bar{1} \\ \bar{4}\bar{4}\bar{4}\bar{0}\bar{0} \end{array}, \quad \begin{array}{l} 034\bar{3}\bar{2}\bar{1} \\ \bar{4}\bar{4}\bar{4}\bar{0}\bar{0} \end{array}, \quad \begin{array}{l} 034\bar{2}\bar{1} \\ \bar{4}\bar{4}\bar{3}\bar{0}\bar{0} \end{array}, \quad \begin{array}{l} 044\bar{2}\bar{1} \\ \bar{4}\bar{4}\bar{0}\bar{0} \end{array}, \quad \begin{array}{l} 044\bar{2}\bar{1} \\ \bar{4}\bar{0}\bar{0} \end{array}, \quad \begin{array}{l} 144\bar{2}\bar{1} \\ \bar{1}\bar{0} \end{array}, \quad \begin{array}{l} 144\bar{2}\bar{1} \\ \bar{0} \end{array}.$$

Adding the ($\boxed{0}$, $\boxed{\bar{0}}$) pair $z = 1$ time, we get

$$\mathbb{T}(b'_2) = 0144\bar{2}\bar{1}\bar{0}, \qquad \mathbb{T}(b'_1) = 00\bar{4}\bar{4}\bar{4}\bar{4}\bar{1}\bar{0}\bar{0}.$$

Therefore we obtain

$$b'_1 = (0,\ldots,0,4,0,0,1) \in B_9, \quad b'_2 = (1,0,0,2,0,\ldots,0,1,1) \in B_7.$$

4 Proof : $C_n^{(1)}$ case

We call an element b of a $U'_q(C_n^{(1)})$ crystal a $U_q(C_n)$ *highest element* if it satisfies $\tilde{e}_i b = 0$ for $i = 1, 2, \ldots, n$. Let $b'_2 \otimes b'_1 = \iota(b_1 \otimes b_2)$ under the isomorphism of $U'_q(C_n^{(1)})$ crystals $\iota : B_l \otimes B_k \xrightarrow{\sim} B_k \otimes B_l$. By definition if

$b_1 \otimes b_2$ is a $U_q(C_n)$ highest element so is $b'_2 \otimes b'_1$. In Section 4.1 we prove Proposition 4.1. It verifies Theorem 3.4 when $b_1 \otimes b_2$ is a $U_q(C_n)$ highest element and either $\mathbb{T}(b_1)$ or $\mathbb{T}(b_2)$ is free of $\boxed{0}$. In Section 4.2 we prove that if both $\mathbb{T}(b_1)$ and $\mathbb{T}(b_2)$ have at least one $\boxed{0}$, then the combinatorial R on $B_l \otimes B_k$ is reduced to the combinatorial R on $B_{l-2} \otimes B_{k-2}$ by removing a ($\boxed{0}$, $\boxed{\overline{0}}$) pair. In Section 4.3 we quote a proposition [B] that assures the compatibility of the column insertion algorithm with a $U_q(C_n)$ crystal isomorphism. Based on these preparations we complete the proof of Theorem 3.4 for general elements in Section 4.4.

4.1 Combinatorial R for a class of highest elements

Proposition 4.1. *Given $b_1 \otimes b_2 \in B_l \otimes B_k$, let $b'_2 \otimes b'_1 = \iota(b_1 \otimes b_2) \in B_k \otimes B_l$ be the image under the isomorphism. Suppose that $b_1 \otimes b_2$ is a $U_q(C_n)$ highest element, and $\mathbb{T}(b_1)$ or $\mathbb{T}(b_2)$ has no $\boxed{0}$. Then $\mathbb{T}(b'_2)$ or $\mathbb{T}(b'_1)$ also has no $\boxed{0}$, and their column insertions give a common tableau:*

$$(\mathbb{T}(b_2) \longrightarrow \mathbb{T}(b_1)) = (\mathbb{T}(b'_1) \longrightarrow \mathbb{T}(b'_2))\,. \tag{4.1}$$

The value of the energy function is given by

$$H(b_1 \otimes b_2) = (\mathbb{T}(b_2) \longrightarrow \mathbb{T}(b_1))_1 - \max(l, k).$$

We give a proof of Proposition 4.1 by a case checking in Appendix A. In this subsection we only list up all the $U_q(C_n)$ highest elements of the above type. We also list up the values of their energy functions. Let $(x_1, x_2, \text{—}, \overline{x}_1)$ stand for $(x_1, x_2, 0, \ldots, 0, \overline{x}_1)$.

Lemma 4.2. *We have*

$$\iota : (l, 0, \text{—}, 0) \otimes (k, 0, \text{—}, 0) \mapsto (k, 0, \text{—}, 0) \otimes (l, 0, \text{—}, 0)$$

under the isomorphism $\iota : B_l \otimes B_k \xrightarrow{\sim} B_k \otimes B_l$.

Proof. They are the unique elements in $B_l \otimes B_k$ and $B_k \otimes B_l$ respectively that do not vanish when $(\tilde{e}_0)^{l+k}$ is applied. ◻

Lemma 4.3. *Let $b_1 \otimes b_2 \in B_l \otimes B_k$ $(l \geq k)$. Suppose that $b_1 \otimes b_2$ is a $U_q(C_n)$ highest element, and $\mathbb{T}(b_1)$ or $\mathbb{T}(b_2)$ has no $\boxed{0}$. Then it has either the form:*

$$(l, 0, \text{—}, 0) \otimes (x_1, x_2, \text{—}, \overline{x}_1)$$

with $x_1, x_2, \overline{x}_1 \in \mathbb{Z}_{\geq 0}$ and $x_1 + x_2 + \overline{x}_1 \leq k$, or the form:

$$(l - 2y_0, 0, \text{—}, 0) \otimes (x_1, x_2, \text{—}, k - x_1 - x_2)$$

with $y_0 (\neq 0), x_1, x_2 \in \mathbb{Z}_{\geq 0}$, $l - k \geq 2y_0 - x_1$ and $x_1 + x_2 \leq k$.

We call the former a *type I*, and the latter a *type II* $U_q(C_n)$ highest element. They are exclusive.

Proof. Let $b_1 = (y_1,\dots,y_n,\overline{y}_n,\dots,\overline{y}_1)$ and $b_2 = (x_1,\dots,x_n,\overline{x}_n,\dots,\overline{x}_1)$. Since $b_1\otimes b_2$ is a $U_q(C_n)$ highest element, $\varepsilon_i(b_1\otimes b_2) = \max(\varepsilon_i(b_1),\varepsilon_i(b_1)+\varepsilon_i(b_2)-\varphi_i(b_1)) = 0$ for $i=1,\dots,n$. It means that $\varepsilon_i(b_1)=0$ and $\varphi_i(b_1)\geq\varepsilon_i(b_2)$ for $i=1,\dots,n$. Thus we have $\varepsilon_n(b_1)=\overline{y}_n=0$, and then $\varepsilon_{n-1}(b_1)=\overline{y}_{n-1}+(y_n-\overline{y}_n)_+=0$, i.e. $\overline{y}_{n-1}=y_n=0$. Repeating the same process we come to $\varepsilon_1(b_1)=\overline{y}_1+(y_2-\overline{y}_2)_+=0$, i.e. $\overline{y}_1=y_2=0$. Thus $b_1=(l-2y_0,0,—,0)$ and $\varphi_i(b_1)=(l-2y_0)\delta_{i,1}$. Thus we have $\varepsilon_n(b_2)=\overline{x}_n=0$, and then $\varepsilon_{n-1}(b_2)=\overline{x}_{n-1}+(x_n-\overline{x}_n)_+=0$, i.e. $\overline{x}_{n-1}=x_n=0$. Repeating the same process we come to $\varepsilon_2(b_2)=\overline{x}_2+(x_3-\overline{x}_3)_+=0$, i.e. $\overline{x}_2=x_3=0$. Thus $b_2=(x_1,x_2,—,\overline{x}_1)$ and $\varepsilon_1(b_2)=x_2+\overline{x}_1$. Therefore we have a condition $\varphi_1(b_1)=l-2y_0\geq x_2+\overline{x}_1$. If $\mathbb{T}(b_1)$ has no $\boxed{0}$ then $y_0=0$ and this condition certainly holds. If $\mathbb{T}(b_2)$ has no $\boxed{0}$ then $x_2+\overline{x}_1=k-x_1$, thus we impose the condition $l-k\geq 2y_0-x_1$. □

Lemma 4.4. *Under the isomorphism of $U'_q(C_n^{(1)})$ crystals*

$$\iota : B_l\otimes B_k \overset{\sim}{\to} B_k\otimes B_l \quad (l\geq k),$$

the type I $U_q(C_n)$ highest element is mapped as

$$\begin{aligned}&(l,0,—,0)\otimes(x_1,x_2,—,\overline{x}_1)\\ &\mapsto (k,0,—,0)\otimes(x_1+l-k-y,x_2,—,\overline{x}_1-y),\end{aligned}$$

where $y=\min[l-k,(\overline{x}_1-x_1)_+]$. The value of the energy function for this element is $x_0+(x_1-\overline{x}_1)_+$ with $x_0=(k-x_1-x_2-\overline{x}_1)/2$.

Proof. For a set of operators $\mathcal{O}_1,\mathcal{O}_2,\dots$ on the $U'_q(C_n^{(1)})$ crystals, we define $\prod_i^{2\swarrow\nwarrow 1}\mathcal{O}_i$ by $\mathcal{O}_2\mathcal{O}_1$ for $n=2$, $\mathcal{O}_2\mathcal{O}_3\mathcal{O}_2\mathcal{O}_1$ for $n=3$, $\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\mathcal{O}_3\mathcal{O}_2\mathcal{O}_1$ for $n=4$ and so on. Let $l=2m$ or $l=2m-1$. The lemma can be proved by applying the following sequence of operators

$$\tilde{f}_0^{m+x_0+(x_1-\overline{x}_1)_+}\tilde{e}_1^{\min(x_1,\overline{x}_1)}\left(\prod_i^{2\swarrow\nwarrow 1}\tilde{e}_i^{x_2+\min(x_1,\overline{x}_1)}\right)\tilde{e}_0^{k+m} \tag{4.2}$$

to the both sides of Lemma 4.2.

In the sequel we will show

$$H((l,0,—,0)\otimes(x_1,x_2,—,\overline{x}_1)) = H((l,0,—,0)\otimes(k,0,—,0))-k+x_0+(x_1-\overline{x}_1)_+.$$

In the case $l = 2m$ and $m \geq k$, the value of the energy function was lowered by k when the first to the k-th $\tilde{e}_0$'s were applied, and raised by $x_0+(x_1-\overline{x}_1)_+$ when the $(m+1)$-th to the last $\tilde{f}_0$'s were applied. In the case $l = 2m$ and $m < k$, in addition to the same change as in the previous case, the value of the energy function was raised by $k-m$ when the $(2m+1)$-th to the last $\tilde{e}_0$'s were applied, and lowered by the same amount when the first to the $(k-m)$-th $\tilde{f}_0$'s were applied.

In the case $l = 2m-1$ and $m-1 > k$, the value of the energy function was lowered by k when the first to the k-th $\tilde{e}_0$'s were applied, and raised by $x_0+(x_1-\overline{x}_1)_+$ when the $(m+1)$-th to the last $\tilde{f}_0$'s were applied. In the case $l = 2m-1$ and $m-1 \leq k$, in addition to the same change as the previous case, the value of the energy function was raised by $k-m+1$ when the $2m$-th to the last $\tilde{e}_0$'s were applied, and lowered by the same amount when the first to the $(k-m+1)$-th $\tilde{f}_0$'s were applied.

Recall that we have normalized the energy function as $H_{B_lB_k}((l,0,—,0)\otimes(0,k,—,0)) = 0$. Thus we have $H((l,0,—,0)\otimes(x_1,x_2,—,\overline{x}_1)) = x_0+(x_1-\overline{x}_1)_+$. $\square$

Corollary 4.5. *For any $l,k \in \mathbb{Z}_{\geq 0}$ we have*

$$H_{B_lB_k}((l,0,—,0)\otimes(k,0,—,0)) = \min(l,k).$$

Lemma 4.6. *Under the isomorphism of $U_q'(C_n^{(1)})$ crystals*

$$\iota : B_l \otimes B_k \xrightarrow{\sim} B_k \otimes B_l \quad (l \geq k),$$

the type II $U_q(C_n)$ highest element is mapped as

$$b_1 \otimes b_2 := (l-2y_0,0,—,0)\otimes(x_1,x_2,—,k-x_1-x_2) \mapsto b_2' \otimes b_1',$$

where $b_2' \otimes b_1'$ is given by the following 1 and 2. Let $\overline{x}_1 = k-x_1-x_2$.

1. *If $l-k > y_0 \geq x_1-\overline{x}_1$,*

$$b_2' \otimes b_1' = (k,0,—,0)\otimes(x_1+l-k-y_0-z,x_2,—,\overline{x}_1+y_0-z),$$

where $z = \min[y_0+\overline{x}_1-x_1, l-k-y_0]$. $H(b_1\otimes b_2) = 0$.

2. *If $l-k \leq y_0$ or $y_0 < x_1-\overline{x}_1$,*

$$b_2' \otimes b_1' = (k-2y_0+2w,0,—,0)\otimes(x_1+l-k-w,x_2,—,\overline{x}_1+w),$$

where $w = \min[l-k,(2y_0-x_1+\overline{x}_1)_+]$. $H(b_1\otimes b_2) = \max[y_0-l+k, x_1-\overline{x}_1-y_0]$.

Proof. If $y_0 \geq x_1 - \overline{x}_1$, let $l = 2m$ or $l = 2m - 1$. The lemma can be proved by applying the following sequence of operators

$$\tilde{f}_0^{m-y_0} \tilde{e}_1^{x_1} \left(\prod_i^{2 \swarrow \nwarrow 1} \tilde{e}_i^{x_2+x_1} \right) \tilde{e}_0^{k+m} \tag{4.3}$$

to the both sides of Lemma 4.2. In the case $l = 2m$ and $m \geq k$, the value of the energy function was lowered by k when the first to the k-th $\tilde{e}_0$'s were applied. In the case $l = 2m$, $m < k$ and $2m - k > y_0$ (resp. $2m - k \leq y_0$), in addition to the same change in the previous case, the value of the energy function was raised by $k - m$ when the $(2m + 1)$-th to the last $\tilde{e}_0$'s were applied, and lowered by $k - m$ (resp. $m - y_0$) when the first to the $(k - m)$-th (resp. the last) $\tilde{f}_0$'s were applied. In the case $l = 2m - 1$ and $m - 1 > k$, the value of the energy function was lowered by k when the first to the k-th $\tilde{e}_0$'s were applied. In the case $l = 2m - 1$, $m - 1 \leq k$ and $2m - 1 - k > y_0$ (resp. $2m - 1 - k \leq y_0$), in addition to the same change in the previous case, the value of the energy function was raised by $k - m + 1$ when the $2m$-th to the last $\tilde{e}_0$'s were applied, and lowered by $k - m + 1$ (resp. $m - y_0$) when the first to the $(k - m + 1)$-th (resp. the last) $\tilde{f}_0$'s were applied.

If $y_0 < x_1 - \overline{x}_1 \leq 2y_0$, one can check that $\tilde{e}_0^{x_1 - \overline{x}_1 - y_0}(b_1 \otimes b_2) = b_1 \otimes \tilde{e}_0^{x_1 - \overline{x}_1 - y_0} b_2$. Lemma 4.7 and the previous case of the present lemma enable us to obtain its image under the map ι. They also tell us that now the value of the energy function is equal to $(2y_0 - x_1 + \overline{x}_1 + k - l)_+$. Then apply $\tilde{f}_0^{x_1 - \overline{x}_1 - y_0}$. Since it again turns out to hit the right component of the tensor product, the value of the energy function is raised by $x_1 - \overline{x}_1 - y_0$.

If $2y_0 < x_1 - \overline{x}_1$, one can check that $\tilde{e}_0^{x_1 - \overline{x}_1 - y_0}(b_1 \otimes b_2) = b_1 \otimes \tilde{e}_0^{x_1 - \overline{x}_1 - y_0} b_2$. Lemma 4.7 and 4.4 enable us to obtain its image under the map ι. They also tell us that now the value of the energy function is equal to 0. Then apply $\tilde{f}_0^{x_1 - \overline{x}_1 - y_0}$. Since it again hits the right component of the tensor product, the value of the energy function is raised by $x_1 - \overline{x}_1 - y_0$. ☐

4.2 Relation of R on $B_l \otimes B_k$ and $B_{l-2} \otimes B_{k-2}$

Let $l \geq 3$. For any $b = (x_1, \ldots, \bar{x}_1) \in B_{l-2}$ we define $\tau^l_{l-2}(b) \in B_l$ to be the unique element such that the tableau $\mathbb{T}(\tau^l_{l-2}(b))$ is made from the tableau $\mathbb{T}(b)$ by adding a ($\boxed{0}$, $\boxed{\overline{0}}$) pair. Note that $\tau^l_{l-2}(b)$ also has the same presentation $(x_1, \ldots, \bar{x}_1)$ in B_l. The map $\tau^l_{l-2} : B_{l-2} \to B_l$ is injective and has the property:

$$\begin{aligned} \tilde{f}_i \tau^l_{l-2}(b) &= \tau^l_{l-2}(\tilde{f}_i b) \quad (0 \leq i \leq n) \qquad \text{if } \tilde{f}_i b \neq 0, \\ \tilde{e}_i \tau^l_{l-2}(b) &= \tau^l_{l-2}(\tilde{e}_i b) \quad (0 \leq i \leq n) \qquad \text{if } \tilde{e}_i b \neq 0. \end{aligned} \tag{4.4}$$

Lemma 4.7. *We have $\tau^l_{l-2}(b_1) \otimes \tau^k_{k-2}(b_2) \simeq \tau^k_{k-2}(b'_2) \otimes \tau^l_{l-2}(b'_1)$ under the isomorphism $B_l \otimes B_k \simeq B_k \otimes B_l$, if and only if $b_1 \otimes b_2 \simeq b'_2 \otimes b'_1$ under $B_{l-2} \otimes B_{k-2} \simeq B_{k-2} \otimes B_{l-2}$. We also have $H_{B_l B_k}(\tau^l_{l-2}(b_1) \otimes \tau^k_{k-2}(b_2)) = H_{B_{l-2}B_{k-2}}(b_1 \otimes b_2)$.*

Proof. Since τ^l_{l-2} and τ^k_{k-2} are injective, the *only if* part of the statement follows immediately after when the *if* part is proved. Without loss of generality we assume $l \geq k$. Set

$$b^{(l)} = (l, 0, —, 0) \in B_l.$$

First consider the case $b_1 = b^{(l-2)} \in B_{l-2}$ and $b_2 = b^{(k-2)} \in B_{k-2}$. Then $b'_2 = b^{(k-2)}$, $b'_1 = b^{(l-2)}$ and $H_{B_{l-2}B_{k-2}}(b^{(l-2)} \otimes b^{(k-2)}) = k-2$ by Corollary 4.5. On the other hand we have

$$\begin{aligned} \psi\left(b^{(l)} \otimes b^{(k)}\right) &= \tau^l_{l-2}(b^{(l-2)}) \otimes \tau^k_{k-2}(b^{(k-2)}) \in B_l \otimes B_k, \\ \psi\left(b^{(k)} \otimes b^{(l)}\right) &= \tau^k_{k-2}(b^{(k-2)}) \otimes \tau^l_{l-2}(b^{(l-2)}) \in B_k \otimes B_l, \end{aligned}$$

where

$$\begin{aligned} \psi &= \tilde{e}_0 (\tilde{e}_1)^{l+k-2} (\tilde{e}_2)^{l+k-2} \cdots (\tilde{e}_{n-1})^{l+k-2} (\tilde{e}_n)^{l+k-2} \\ &\quad \times (\tilde{e}_{n-1})^{l+k-2} \cdots (\tilde{e}_2)^{l+k-2} (\tilde{e}_1)^{l+k-2} (\tilde{e}_0)^{l+k-1}. \end{aligned}$$

By Lemma 4.2 one has

$$\tau^l_{l-2}(b^{(l-2)}) \otimes \tau^k_{k-2}(b^{(k-2)}) \simeq \tau^k_{k-2}(b^{(k-2)}) \otimes \tau^l_{l-2}(b^{(l-2)}) \tag{4.5}$$

under the isomorphism $B_l \otimes B_k \simeq B_k \otimes B_l$. The energy was lowered by k when the first to the k-th $\tilde{e}_0$'s were applied, and raised by $k-1$ when the $(l+1)$-th to the $(l+k-1)$-th $\tilde{e}_0$'s were applied. Then it was lowered by 1 when the leftmost $\tilde{e}_0$ was applied. Thus we have

$$\begin{aligned} H_{B_l B_k}(\tau^l_{l-2}(b^{(l-2)}) \otimes \tau^k_{k-2}(b^{(k-2)})) &= H_{B_l B_k}(b^{(l)} \otimes b^{(k)}) - 2 = k-2 \\ &= H_{B_{l-2}B_{k-2}}(b^{(l-2)} \otimes b^{(k-2)}). \end{aligned} \tag{4.6}$$

The proof is finished in this special case from Corollary 4.5.

Now we consider the general elements $b_1 \otimes b_2 \in B_{l-2} \otimes B_{k-2}$ and $b'_2 \otimes b'_1 \in B_{k-2} \otimes B_{l-2}$ that are mapped to each other under the isomorphism. Take any finite sequence ψ' made of $\tilde{e}_i$'s and $\tilde{f}_i$'s $(i = 0, 1, \ldots, n)$ such that

$$b_1 \otimes b_2 = \psi'(b^{(l-2)} \otimes b^{(k-2)}), \tag{4.7}$$

which is equivalent to

$$b'_2 \otimes b'_1 = \psi'(b^{(k-2)} \otimes b^{(l-2)}). \tag{4.8}$$

For any operator in ψ', the rules (2.1)-(2.2) determine whether it should hit the left or the right component of the tensor product. For any $c_1 \otimes c_2 \in B_{l-2} \otimes B_{k-2}$ we have $\varphi_i(\tau^l_{l-2}(c_1)) = \varphi_i(c_1) + \delta_{i,0}$ and $\varepsilon_i(\tau^k_{k-2}(c_2)) = \varepsilon_i(c_2) + \delta_{i,0}$ from (2.8). Thus the alternatives in (2.1)-(2.2) are not changed by $\tau^l_{l-2} \otimes \tau^k_{k-2}$. From (4.4) it follows that $(\tau^l_{l-2} \otimes \tau^k_{k-2})(\psi'(c_1 \otimes c_2)) = \psi'(\tau^l_{l-2}(c_1) \otimes \tau^k_{k-2}(c_2))$. Applying $\tau^l_{l-2} \otimes \tau^k_{k-2}$ (resp. $\tau^k_{k-2} \otimes \tau^l_{l-2}$) to (4.7) (resp. (4.8)) we thus get

$$\tau^l_{l-2}(b_1) \otimes \tau^k_{k-2}(b_2) = \psi'(\tau^l_{l-2}(b^{(l-2)}) \otimes \tau^k_{k-2}(b^{(k-2)})), \tag{4.9}$$
$$\tau^k_{k-2}(b'_2) \otimes \tau^l_{l-2}(b'_1) = \psi'(\tau^k_{k-2}(b^{(k-2)}) \otimes \tau^l_{l-2}(b^{(l-2)})). \tag{4.10}$$

From (4.5) it follows that

$$\tau^l_{l-2}(b_1) \otimes \tau^k_{k-2}(b_2) \simeq \tau^k_{k-2}(b'_2) \otimes \tau^l_{l-2}(b'_1)$$

under the isomorphism $B_l \otimes B_k \simeq B_k \otimes B_l$. When comparing (4.7) and (4.9) change of the value of the energy function caused by ψ' is not affected by $\tau^l_{l-2} \otimes \tau^k_{k-2}$. Therefore from (4.6) we have $H_{B_l B_k}(\tau^l_{l-2}(b_1) \otimes \tau^k_{k-2}(b_2)) = H_{B_{l-2} B_{k-2}}(b_1 \otimes b_2)$. □

4.3 Column insertion and $U_q(C_n)$ crystal morphism

The next proposition is due to Baker (Proposition 7.1 of [B]). For a dominant integral weight λ of the C_n root system, let $B(\lambda)$ be the $U_q(C_n)$ crystal associated with the irreducible highest weight representation $V(\lambda)$ [KN]. The elements of $B(\lambda)$ can be represented by the semistandard C-tableaux of shape λ [KN].

Proposition 4.8. *Let $B(\mu) \otimes B(\nu) \simeq \bigoplus_j B(\lambda_j)^{\oplus m_j}$ be the tensor product decomposition of crystals. Here λ_j's are distinct highest weights and $m_j(\geq 1)$ is the multiplicity of $B(\lambda_j)$. Forgetting the multiplicities we have the canonical morphism from $B(\mu) \otimes B(\nu)$ to $\bigoplus_j B(\lambda_j)$. Define ψ_C by*

$$\psi_C(b_1 \otimes b_2) = \left(b_2 \xrightarrow{*} b_1\right).$$

Then ψ_C gives the unique crystal morphism from $B(\mu) \otimes B(\nu)$ to $\bigoplus_j B(\lambda_j)$.

Here $b_2 \xrightarrow{*} b_1$ is the tableau obtained from successive column insertions of letters of the Japanese reading word of b_2 into b_1 by the original definition in [B] (In [B], $b_2 \xrightarrow{*} b_1$ is denoted by $b_1 * b_2$.)

Remark 4.9. The insertion $b_2 \xrightarrow{*} b_1$ may include such a process that $\boxed{x}$ and $\boxed{\overline{x}}$ annihilate pairwise and an empty box thereby produced slides out.

In [B] this process was called a *bumping-sliding transition*. Consider the case that both b_1 and b_2 are one-row tableaux. (We shall omit the symbol $\mathbb{T}$ here.) In this case the bumping-sliding transition can occur only when $\boxed{\overline{1}}$ is inserted into $\boxed{1}$. We defined our column insertion ($\longrightarrow$) so that $\varsigma((b_2 \longrightarrow b_1))$ is equivalent to $(\varsigma(b_2) \xrightarrow{*} \varsigma(b_1))$. In Rule 3.3 for $U'_q(C_n^{(1)})$ combinatorial R matrix we have removed ($\boxed{0}$, $\boxed{\overline{0}}$) pairs beforehand which become ($\boxed{1}$, $\boxed{\overline{1}}$) under ς. Thus we have avoided the bumping-sliding transition to occur.

4.4 Proof of Theorem 3.4

With no loss of generality we assume $l \geq k$. Let $b'_2 \otimes b'_1$ be the image of $b_1 \otimes b_2$ under the isomorphism of $U'_q(C_n^{(1)})$ crystals $\iota : B_l \otimes B_k \xrightarrow{\sim} B_k \otimes B_l$. In order to prove Theorem 3.4 we are to show the claims:

1. Let

   $$z_0 = \min(\sharp\boxed{0} \text{ in } \mathbb{T}(b_1), \sharp\boxed{0} \text{ in } \mathbb{T}(b_2)),\ z'_0 = \min(\sharp\boxed{0} \text{ in } \mathbb{T}(b'_1), \sharp\boxed{0} \text{ in } \mathbb{T}(b'_2)).$$

   Then $z'_0 = z_0$.

2. Remove ($\boxed{0}$,$\boxed{\overline{0}}$) pairs z_0 times from $\mathbb{T}(b_1)$, $\mathbb{T}(b_2)$, $\mathbb{T}(b'_1)$ and $\mathbb{T}(b'_2)$. Call the resulting tableaux $\hat{\mathbb{T}}(b_1)$, $\hat{\mathbb{T}}(b_2)$, $\hat{\mathbb{T}}(b'_1)$ and $\hat{\mathbb{T}}(b'_2)$, respectively. Then we have

   $$\left(\hat{\mathbb{T}}(b_2) \longrightarrow \hat{\mathbb{T}}(b_1)\right) = \left(\hat{\mathbb{T}}(b'_1) \longrightarrow \hat{\mathbb{T}}(b'_2)\right). \tag{4.11}$$

3. $H_{B_l B_k}(b_1 \otimes b_2) = \left(\hat{\mathbb{T}}(b_2) \longrightarrow \hat{\mathbb{T}}(b_1)\right)_1 - \hat{\mathbb{T}}(b_1)_1.$

Proof. Thanks to Lemma 4.7 it suffices to verify the above claims only when $\mathbb{T}(b_1)$ or $\mathbb{T}(b_2)$ has no $\boxed{0}$. Such a case can be reduced to Proposition 4.1 by the argument as follows.

Let $b_1 \otimes b_2$ be an element of $B_l \otimes B_k$, which is not necessarily a $U_q(C_n)$ highest element and either $\mathbb{T}(b_1)$ or $\mathbb{T}(b_2)$ is free of $\boxed{0}$. Let $b'_2 \otimes b'_1 = \iota(b_1 \otimes b_2)$ under the isomorphism $\iota : B_l \otimes B_k \rightarrow B_k \otimes B_l$. There exists a sequence $i_1, i_2, \ldots, i_s$ $(1 \leq i_\alpha \leq n,\ \alpha = 1, 2, \ldots, s)$ such that $\dot{b}_1 \otimes \dot{b}_2 := \tilde{e}_{i_s} \cdots \tilde{e}_{i_1}(b_1 \otimes b_2)$ is a $U_q(C_n)$ highest element. Then we have $b'_2 \otimes b'_1 = \tilde{f}_{i_1} \cdots \tilde{f}_{i_s} \circ \iota \circ \tilde{e}_{i_s} \cdots \tilde{e}_{i_1}(b_1 \otimes b_2)$. Let $\dot{b}'_2 \otimes \dot{b}'_1 = \iota(\dot{b}_1 \otimes \dot{b}_2)$. Since $\tilde{e}_i$ $(1 \leq i \leq n)$ does not change $\sharp\boxed{0}$, $\mathbb{T}(\dot{b}_1)$ or $\mathbb{T}(\dot{b}_2)$ also has no $\boxed{0}$. Therefore from Proposition 4.1 we have

$$\left(\mathbb{T}(\dot{b}_2) \longrightarrow \mathbb{T}(\dot{b}_1)\right) = \left(\mathbb{T}(\dot{b}'_1) \longrightarrow \mathbb{T}(\dot{b}'_2)\right), \tag{4.12}$$

where $\mathbb{T}(\dot{b}'_1)$ or $\mathbb{T}(\dot{b}'_2)$ has no $\boxed{0}$. Since $b'_2 \otimes b'_1 = \tilde{f}_{i_1} \cdots \tilde{f}_{i_s}(\dot{b}'_2 \otimes \dot{b}'_1)$ and $1 \le i_\alpha \le n$, we conclude that $\mathbb{T}(b'_1)$ or $\mathbb{T}(b'_2)$ has no $\boxed{0}$. Thus Claim 1 is indeed valid as $z_0 = z'_0 = 0$. By Remark 3.2, (4.12) is equivalent to

$$\left(\varsigma(\mathbb{T}(\dot{b}_2)) \xrightarrow{*} \varsigma(\mathbb{T}(\dot{b}_1))\right) = \left(\varsigma(\mathbb{T}(\dot{b}'_1)) \xrightarrow{*} \varsigma(\mathbb{T}(\dot{b}'_2))\right) =: \mathbb{P}. \tag{4.13}$$

Here ς is defined in (2.6). Regarding $\mathbb{P}$ as an element of a $U_q(C_{n+1})$ crystal, we apply Proposition 4.8, to get

$$\left(\varsigma(\mathbb{T}(b_2)) \xrightarrow{*} \varsigma(\mathbb{T}(b_1))\right) = \tilde{f}_{i_1+1} \cdots \tilde{f}_{i_s+1}(\mathbb{P}) = \left(\varsigma(\mathbb{T}(b'_1)) \xrightarrow{*} \varsigma(\mathbb{T}(b'_2))\right), \tag{4.14}$$

which is equivalent to

$$(\mathbb{T}(b_2) \longrightarrow \mathbb{T}(b_1)) = (\mathbb{T}(b'_1) \longrightarrow \mathbb{T}(b'_2)), \tag{4.15}$$

showing Claim 2. Since $\tilde{f}_i \, (1 \le i \le n)$ does not change the shape of the tableaux [B] and $H(b_1 \otimes b_2) = H(\dot{b}_1 \otimes \dot{b}_2)$, Claim 3 follows from Proposition 4.1. ◻

5 $U'_q(A^{(2)}_{2n-1})$ crystal case

5.1 Definitions

Given a non-negative integer l, let us denote by B_l the $U'_q(A^{(2)}_{2n-1})$ crystal defined in [KKM]. (Their B_l is identical to our B_l.) As a set B_l reads

$$B_l = \left\{ (x_1, \ldots, x_n, \overline{x}_n, \ldots, \overline{x}_1) \;\middle|\; x_i, \overline{x}_i \in \mathbb{Z}_{\ge 0}, \sum_{i=1}^{n} (x_i + \overline{x}_i) = l \right\}.$$

B_l is isomorphic to $B(l\Lambda_1)$ as a crystal for $U_q(C_n)$. The crystal structure is given by

$$\begin{aligned}
\tilde{e}_0 b &= \begin{cases} (x_1, x_2 - 1, \ldots, \overline{x}_2, \overline{x}_1 + 1) & \text{if } x_2 > \overline{x}_2, \\ (x_1 - 1, x_2, \ldots, \overline{x}_2 + 1, \overline{x}_1) & \text{if } x_2 \le \overline{x}_2, \end{cases} \\
\tilde{e}_n b &= (x_1, \ldots, x_n + 1, \overline{x}_n - 1, \ldots, \overline{x}_1), \\
\tilde{e}_i b &= \begin{cases} (x_1, \ldots, x_i + 1, x_{i+1} - 1, \ldots, \overline{x}_1) & \text{if } x_{i+1} > \overline{x}_{i+1}, \\ (x_1, \ldots, \overline{x}_{i+1} + 1, \overline{x}_i - 1, \ldots, \overline{x}_1) & \text{if } x_{i+1} \le \overline{x}_{i+1}, \end{cases} \\
\tilde{f}_0 b &= \begin{cases} (x_1, x_2 + 1, \ldots, \overline{x}_2, \overline{x}_1 - 1) & \text{if } x_2 \ge \overline{x}_2, \\ (x_1 + 1, x_2, \ldots, \overline{x}_2 - 1, \overline{x}_1) & \text{if } x_2 < \overline{x}_2, \end{cases}
\end{aligned}$$

$$\begin{aligned}
\tilde{f}_n b &= (x_1,\dots,x_n-1,\overline{x}_n+1,\dots,\overline{x}_1),\\
\tilde{f}_i b &= \begin{cases}(x_1,\dots,x_i-1,x_{i+1}+1,\dots,\overline{x}_1) & \text{if } x_{i+1}\ge\overline{x}_{i+1},\\ (x_1,\dots,\overline{x}_{i+1}-1,\overline{x}_i+1,\dots,\overline{x}_1) & \text{if } x_{i+1}<\overline{x}_{i+1},\end{cases}
\end{aligned}\tag{5.1}$$

where $b=(x_1,\dots,x_n,\overline{x}_n,\dots,\overline{x}_1)$ and $i=1,\dots,n-1$. For this b we have

$$\begin{aligned}
\varphi_0(b) &= \overline{x}_1+(\overline{x}_2-x_2)_+,\quad \varepsilon_0(b)=x_1+(x_2-\overline{x}_2)_+,\\
\varphi_i(b) &= x_i+(\overline{x}_{i+1}-x_{i+1})_+ \quad \text{for}\, i=1,\dots,n-1,\\
\varepsilon_i(b) &= \overline{x}_i+(x_{i+1}-\overline{x}_{i+1})_+ \quad \text{for}\, i=1,\dots,n-1,\\
\varphi_n(b) &= x_n,\quad \varepsilon_n(b)=\overline{x}_n.
\end{aligned}\tag{5.2}$$

We shall depict the element $b=(x_1,\dots,x_n,\overline{x}_n,\dots,\overline{x}_1)\in B_l$ with the tableau:

$$\mathcal{T}(b)=\boxed{\overbrace{1\cdots1}^{x_1}}\boxed{\cdots}\boxed{\overbrace{n\cdots n}^{x_n}}\boxed{\overbrace{\overline{n}\cdots\overline{n}}^{\overline{x}_n}}\boxed{\cdots}\boxed{\overbrace{\overline{1}\cdots\overline{1}}^{\overline{x}_1}}\,. \tag{5.3}$$

The length of this one-row tableau is equal to l, namely $\sum_{i=1}^n(x_i+\overline{x}_i)=l$.

In this section we normalize the energy function as

$$H_{B_lB_k}((l,0,—,0)\otimes(0,—,0,k))=0, \tag{5.4}$$

irrespective of $l<k$ or $l\ge k$.

5.2 Main theorem : $A^{(2)}_{2n-1}$ case

The insertion symbol $\longrightarrow$ and the reverse bumping in this section are the same ones as in Section 3.

Given $b_1\otimes b_2\in B_l\otimes B_k$, we define the element $b_2'\otimes b_1'\in B_k\otimes B_l$ and $l',k',m\in\mathbb{Z}_{\ge0}$ by the following rule.

Rule 5.1.
Set $z=\min(\sharp\boxed{1}\text{ in }\mathcal{T}(b_1),\ \sharp\boxed{\overline{1}}\text{ in }\mathcal{T}(b_2))$. Remove $\boxed{1}$'s (resp. $\boxed{\overline{1}}$'s) from $\mathcal{T}(b_1)$ (resp. $\mathcal{T}(b_2)$) z times and call the resulting tableaux $\acute{\mathcal{T}}(b_1)$ (resp.$\grave{\mathcal{T}}(b_2)$). Let $l'=\acute{\mathcal{T}}(b_1)_1=l-z$ and $k'=\grave{\mathcal{T}}(b_2)_1=k-z$. Operate the column insertion and set $\mathbb{P}(b_2\xrightarrow{*}b_1)=\left(\grave{\mathcal{T}}(b_2)\longrightarrow\acute{\mathcal{T}}(b_1)\right)$. (This $\mathbb{P}(b_2\xrightarrow{*}b_1)$ coincides with the column insertion $\mathcal{T}(b_2)\xrightarrow{*}\mathcal{T}(b_1)$, because of $\left(\boxed{\overline{1}}\xrightarrow{*}\boxed{1}\right)=\emptyset$.) $\mathbb{P}(b_2\xrightarrow{*}b_1)$ has the form:

$$\begin{array}{|c|c|}\hline j_1\cdots\cdots j_{k'} & i_{m+1}\ \cdots\ i_{l'}\\ \hline i_1\cdots i_m & \\ \cline{1-1}\end{array}$$

where m is the length of the second row, hence that of the first row is $l' + k' - m$. $(0 \le m \le k'.)$

Next we bump out l' letters from the tableau $T^{(0)} = \mathbb{P}(b_2 \stackrel{*}{\to} b_1)$ by the reverse bumping algorithm. For the boxes containing $i_{l'}, i_{l'-1}, \ldots, i_1$ in the above tableau, we do it first for $i_{l'}$ then $i_{l'-1}$ and so on. Correspondingly, let w_1 be the first letter that is bumped out from the leftmost column and w_2 be the second and so on. Denote by $T^{(i)}$ the resulting tableau when w_i is bumped out $(1 \le i \le l')$. Now $b'_1 \in B_l$ and $b'_2 \in B_k$ are uniquely specified by

$$\mathcal{T}(b'_2) = \begin{array}{|c|c|}\multicolumn{1}{c}{z} & \multicolumn{1}{c}{} \\ \hline 1\cdots 1 & \quad T^{(l')} \quad \\ \hline \end{array} \;, \quad \mathcal{T}(b'_1) = \begin{array}{|c|c|c|c|}\multicolumn{3}{c}{} & \multicolumn{1}{c}{z} \\ \hline w_1 & \cdots & w_{l'} & \bar{1}\cdots\bar{1} \\ \hline \end{array}\;.$$

Our main result for $U'_q(A^{(2)}_{2n-1})$ is

Theorem 5.2. *Given $b_1 \otimes b_2 \in B_l \otimes B_k$, specify $b'_2 \otimes b'_1 \in B_k \otimes B_l$ and l', k', m by Rule 5.1. Let $\iota : B_l \otimes B_k \stackrel{\sim}{\to} B_k \otimes B_l$ be the isomorphism of $U'_q(A^{(2)}_{2n-1})$ crystal. Then we have*

$$\iota(b_1 \otimes b_2) = b'_2 \otimes b'_1,$$
$$H_{B_l B_k}(b_1 \otimes b_2) = 2\min(l', k') - m.$$

Example 5.3. If $1123 \otimes 1\bar{1}\bar{1}$ is regarded as an element of $U'_q(A^{(2)}_{2n-1})$ crystal $B_4 \otimes B_3$, it is mapped to $113 \otimes 12\bar{1}\bar{1} \in B_3 \otimes B_4$ under the isomorphism. Here $\mathbb{P}(1\bar{1}\bar{1} \stackrel{*}{\to} 1123) = 123$ and $H(1123 \otimes 1\bar{1}\bar{1}) = 2$. If $1123 \otimes 1\bar{1}\bar{1}$ is regarded as an element of $U'_q(C^{(1)}_n)$ crystal $B_4 \otimes B_3$, it is mapped to $123 \otimes 01\bar{1}\bar{0} \in B_3 \otimes B_4$ under the isomorphism. Here $\hat{\mathbb{P}}(1\bar{1}\bar{1} \to 1123) = {\scriptsize\begin{matrix} 0 & 1 & 2 & 3 \\ 1 & \bar{1} & \bar{0} & \end{matrix}}$ and $H(1123 \otimes 1\bar{1}\bar{1}) = 0$.

5.3 Proof : $A^{(2)}_{2n-1}$ case

Given $b_1 \otimes b_2 \in B_k \otimes B_k$, determine $b'_2 \otimes b'_1 \in B_k \otimes B_l$ by Rule 5.1. To prove Theorem 5.2, we are to show the following claims:

$$1. \quad \left(\grave{\mathcal{T}}(b_2) \longrightarrow \acute{\mathcal{T}}(b_1)\right) = \left(\grave{\mathcal{T}}(b'_1) \longrightarrow \acute{\mathcal{T}}(b'_2)\right). \tag{5.5}$$

$$2. \quad H_{B_l B_k}(b_1 \otimes b_2) = \left(\grave{\mathcal{T}}(b_2) \longrightarrow \acute{\mathcal{T}}(b_1)\right)_1 - |l - k|. \tag{5.6}$$

Lemma 5.4. *We have*

$$\iota : (l, 0, —, 0) \otimes (k, 0, —, 0) \mapsto (k, 0, —, 0) \otimes (l, 0, —, 0)$$

under the isomorphism $B_l \otimes B_k \stackrel{\sim}{\to} B_k \otimes B_l$.

Proof. They are the unique elements in $B_l \otimes B_k$ and $B_k \otimes B_l$ respectively that do not vanish when $(\tilde{e}_0)^{l+k}$ is applied and do not vanish when $(\tilde{f}_1)^{l+k}$ is applied. $\square$

Proof of Theorem 5.2. Claim 1 is due to Proposition 4.8 and the fact that the irreducible decomposition of the $U_q(C_n)$ module $V(l\Lambda_1) \otimes V(k\Lambda_1)$ is multiplicity-free (for generic q).

We call an element b of a $U_q'(A^{(2)}_{2n-1})$ crystal a $U_q(C_n)$ *highest element* if it satisfies $\tilde{e}_i b = 0$ for $i = 1, 2, \ldots, n$. To show Claim 2, it suffices to check it for $U_q(C_n)$ highest elements. Then the general case follows from Proposition 4.8 because $H(\tilde{f}_{i_1} \cdots \tilde{f}_{i_j}(b_1 \otimes b_2)) = H(b_1 \otimes b_2)$ for $i_1, \ldots, i_j \in \{1, \ldots, n\}$ if $\tilde{f}_{i_1} \cdots \tilde{f}_{i_j}(b_1 \otimes b_2) \neq 0$. We assume $l \geq k$ with no loss of generality. Suppose that $b_1 \otimes b_2 \simeq b_2' \otimes b_1'$ is a $U_q(C_n)$ highest element. In general it has the form:

$$b_1 \otimes b_2 = (l, 0, —, 0) \otimes (x_1, x_2, —, \overline{x}_1),$$

where x_1, x_2 and $\overline{x}_1$ are arbitrary as long as $k = x_1 + x_2 + \overline{x}_1$. Applying

$$\tilde{e}_0^{\overline{x}_1} \tilde{e}_2^{x_2+\overline{x}_1} \cdots \tilde{e}_{n-1}^{x_2+\overline{x}_1} \tilde{e}_n^{x_2+\overline{x}_1} \tilde{e}_{n-1}^{x_2+\overline{x}_1} \cdots \tilde{e}_2^{x_2+\overline{x}_1} \tilde{e}_0^{x_2+\overline{x}_1}$$

to the both sides of Lemma 5.4, we find

$$(l, 0, —, 0) \otimes (x_1, x_2, —, \overline{x}_1) \simeq (k, 0, —, 0) \otimes (x_1', x_2, —, \overline{x}_1).$$

Here $x_1' = l - x_2 - \overline{x}_1$. In the course of the application of $\tilde{e}_i$'s, the value of the energy function has changed as

$$H\left((l, 0, —, 0) \otimes (x_1, x_2, —, \overline{x}_1)\right) = H\left((l, 0, —, 0) \otimes (k, 0, —, 0)\right) - x_2 - 2\overline{x}_1.$$

Thus according to our normalization (5.4) we have $H(b_1 \otimes b_2) = 2k - x_2 - 2\overline{x}_1$. On the other hand for this highest element the column insertions (5.5) lead to a tableau $\begin{array}{|l|}\hline 1 \cdots\cdots 1 \\ \hline 2 \cdots 2 \\ \hline \end{array}$ whose first row has the length $l + k - x_2 - 2\overline{x}_1$. This completes the proof of Claim 2. $\square$

Remark 5.5. For $b = (x_1, \ldots, \overline{x}_1) \in B_{l-1}$ $(l \geq 2)$ define

$$\begin{aligned} \acute{\tau}^l_{l-1}(b) &= (x_1 + 1, x_2, \ldots, \overline{x}_1) \in B_l, \\ \grave{\tau}^l_{l-1}(b) &= (x_1, \ldots, \overline{x}_2, \overline{x}_1 + 1) \in B_l. \end{aligned}$$

Then we have $\acute{\tau}^l_{l-1}(c_1) \otimes \grave{\tau}^k_{k-1}(c_2) \simeq \acute{\tau}^k_{k-1}(c_2') \otimes \grave{\tau}^l_{l-1}(c_1')$ under the isomorphism $B_l \otimes B_k \simeq B_k \otimes B_l$, if and only if $c_1 \otimes c_2 \simeq c_2' \otimes c_1'$ under $B_{l-1} \otimes B_{k-1} \simeq B_{k-1} \otimes B_{l-1}$. We also have $H_{B_l B_k}(\acute{\tau}^l_{l-1}(c_1) \otimes \grave{\tau}^k_{k-1}(c_2)) = H_{B_{l-1} B_{k-1}}(c_1 \otimes c_2)$.

A Proof of Proposition 4.1

A.1 Column insertions of type I $U_q(C_n)$ highest elements

Let us consider an element in $B_l \otimes B_k$ depicted by

$$b_1 \otimes b_2 = \begin{array}{|c|}\multicolumn{1}{c}{l} \\ \hline \qquad\qquad 1 \qquad\qquad \\ \hline \multicolumn{1}{c}{} \end{array} \otimes \begin{array}{|c|c|c|c|c|}\hline 0 & 1 & 2 & \bar{1} & \bar{0} \\ \hline \multicolumn{1}{c}{x_0} & \multicolumn{1}{c}{x_1} & \multicolumn{1}{c}{x_2} & \multicolumn{1}{c}{\overline{x}_1} & \multicolumn{1}{c}{x_0} \end{array}.$$

(In this appendix we denote $\mathbb{T}(b)$ simply by b.) It is a $U_q(C_n)$ highest element. We denote by $b'_2 \otimes b'_1$ the image of this element under the isomorphism $\iota : B_l \otimes B_k \to B_k \otimes B_l$.

A.1.1

Let $\overline{x}_1 \leq x_1$. Then $b'_2 \otimes b'_1$ is depicted by

$$b'_2 \otimes b'_1 = \begin{array}{|c|}\multicolumn{1}{c}{k} \\ \hline \qquad 1 \qquad \\ \hline \multicolumn{1}{c}{} \end{array} \otimes \begin{array}{|c|c|c|c|c|}\multicolumn{1}{c}{} & \multicolumn{1}{c}{x_1+l-k} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline \quad 0 \quad & \qquad 1 \qquad & \quad 2 \quad & \quad \bar{1} \quad & \quad \bar{0} \quad \\ \hline \multicolumn{1}{c}{x_0} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{x_2} & \multicolumn{1}{c}{\overline{x}_1} & \multicolumn{1}{c}{x_0} \end{array}$$

The column insertions $(b_2 \longrightarrow b_1)$ and $(b'_1 \longrightarrow b'_2)$ lead to the same intermediate result;

$$\begin{array}{|c|c|}\hline \quad 0 \quad & \quad 1 \quad \\ \hline \multicolumn{1}{c}{x_0} & \multicolumn{1}{c}{x_1-\overline{x}_1} \end{array} \longrightarrow \begin{array}{|c|c|c|c}\multicolumn{1}{c}{} & \multicolumn{3}{c}{l-\overline{x}_1} \\ \hline \quad 0 \quad & \multicolumn{3}{c|}{1} \\ \hline 1 & 2 & \multicolumn{1}{c|}{\bar{0}} & \\ \cline{1-3} \multicolumn{1}{c}{\overline{x}_1} & \multicolumn{1}{c}{x_2} & \multicolumn{1}{c}{\overline{x}_1+x_0} & \end{array}$$

The value of the energy function is $x_0 + x_1 - \overline{x}_1$.

A.1.2

Let $\overline{x}_1 > x_1$. Then $b'_2 \otimes b'_1$ is depicted by

$$b'_2 \otimes b'_1 = \begin{array}{|c|}\multicolumn{1}{c}{k} \\ \hline \qquad 1 \qquad \\ \hline \multicolumn{1}{c}{} \end{array} \otimes \begin{array}{|c|c|c|c|c|}\multicolumn{1}{c}{} & \multicolumn{1}{c}{x_1+l-k-y} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline \quad 0 \quad & \qquad 1 \qquad & \quad 2 \quad & \quad \bar{1} \quad & \quad \bar{0} \quad \\ \hline \multicolumn{1}{c}{x_0+y} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{x_2} & \multicolumn{1}{c}{\overline{x}_1-y} & \multicolumn{1}{c}{x_0+y} \end{array}$$

where

$$y = \min[l-k, \overline{x}_1 - x_1]. \tag{A.1}$$

The column insertions $(b_2 \longrightarrow b_1)$ and $(b'_1 \longrightarrow b'_2)$ lead to the same intermediate result;
For $x_1 + x_2 > \overline{x}_1$,

$$\underset{x_0}{\boxed{\quad 0 \quad}} \longrightarrow \begin{array}{l} \overset{\overline{x}_1}{\boxed{\; 0 \;}}\overset{l-\overline{x}_1}{\boxed{\qquad 1 \qquad}} \\ \underset{x_1}{\boxed{1}}\underset{x_2}{\boxed{2}}\underset{\overline{x}_1+x_0}{\boxed{\; \bar{0} \;}} \end{array}$$

For $x_1 + x_2 \leq \overline{x}_1$,

$$\underset{x_0}{\boxed{\quad 0 \quad}} \longrightarrow \begin{array}{l} \overset{\overline{x}_1-x_1-x_2}{\boxed{\; 0 \;}\boxed{0}\!\boxed{1}}\overset{l-\overline{x}_1}{\boxed{\qquad 1 \qquad}} \\ \underset{x_1}{\boxed{1}}\underset{x_2}{\boxed{2}}\boxed{\bar{1}}\!\boxed{\bar{0}}\underset{x_0+x_1+x_2}{\boxed{\; \bar{0} \;}} \end{array}$$

The value of the energy function is x_0. Here and in the following we use the notation:

$$\overset{m}{\begin{array}{|c||c|} \hline 0 & 1 \\ \hline \bar{1} & \bar{0} \\ \hline \end{array}} = \begin{array}{cc} \frac{m}{2} & \frac{m}{2} \\ \begin{array}{|c|} \hline 0 \\ \hline \bar{1} \\ \hline \end{array} & \begin{array}{|c|} \hline 1 \\ \hline \bar{0} \\ \hline \end{array} \end{array} \ (m\text{: even}), \quad \begin{array}{cc} \frac{m-1}{2} & \frac{m+1}{2} \\ \begin{array}{|c|} \hline 0 \\ \hline \bar{1} \\ \hline \end{array} & \begin{array}{|c|} \hline 1 \\ \hline \bar{0} \\ \hline \end{array} \\ \frac{m+1}{2} & \frac{m-1}{2} \end{array} \ (m\text{: odd}).$$

A.2 Column insertions of type II $U_q(C_n)$ highest elements

Let

$$b_1 \otimes b_2 = \underset{y_0}{\boxed{\; 0 \;}}\overset{l-2y_0}{\boxed{\qquad 1 \qquad}}\underset{y_0}{\boxed{\; \bar{0} \;}} \otimes \underset{x_1}{\boxed{1}}\underset{x_2}{\boxed{2}}\underset{\overline{x}_1}{\boxed{\bar{1}}}$$

be a $U_q(C_n)$ highest element in $B_l \otimes B_k$. Thus we assume $l - 2y_0 \geq x_2 + \overline{x}_1$.

A.2.1

Let $l - k > y_0 \geq x_1 - \overline{x}_1$. Then $b_2' \otimes b_1'$ is depicted by

$$b_2' \otimes b_1' = \overset{k}{\boxed{\qquad 1 \qquad}} \otimes \underset{z}{\boxed{0}}\overset{x_1+l-k-y_0-z}{\boxed{\quad 1 \quad}}\underset{x_2}{\boxed{2}}\overset{\overline{x}_1+y_0-z}{\boxed{\bar{1}}}\underset{z}{\boxed{\bar{0}}}$$

where

$$z = \min[y_0 + \overline{x}_1 - x_1, l - k - y_0]. \tag{A.2}$$

The column insertions ($b_2 \longrightarrow b_1$) and ($b_1' \longrightarrow b_2'$) give the same result;
For $y_0 \geq k$,

$$\begin{array}{l} \overset{y_0}{\boxed{\qquad 0 \qquad}}\overset{l-2y_0}{\boxed{1}}\overset{y_0}{\boxed{\qquad \bar{0} \qquad}} \\ \underset{x_1}{\boxed{1}}\underset{x_2}{\boxed{2}}\underset{\overline{x}_1}{\boxed{\bar{1}}} \end{array}$$

For $k > y_0 \geq x_1 + x_2$,

$$\begin{array}{l} \overbrace{\boxed{\quad 0 \quad}}^{y_0}\boxed{0}\,\|\,\boxed{1}\overbrace{\boxed{\ 1\ }}^{l-k-y_0}\overbrace{\boxed{\quad \overline{0} \quad}}^{y_0} \\ \underbrace{\boxed{1}}_{x_1}\underbrace{\boxed{2}}_{x_2}\underbrace{\boxed{\ \overline{1}\ }\boxed{\overline{1}}}_{k-y_0}\,\|\,\boxed{\overline{0}} \end{array}$$

For $x_1 + x_2 > y_0 \geq x_1 + x_2 - \overline{x}_1$,

$$\begin{array}{l} \boxed{\quad 0 \quad}\overbrace{\boxed{0}}^{\overline{x}_1+y_0-x_1-x_2}\,\|\,\boxed{1}\overbrace{\boxed{\quad 1 \quad}}^{l-2y_0-\overline{x}_1}\overbrace{\boxed{\ \overline{0}\ }}^{y_0} \\ \underbrace{\boxed{\ 1\ }}_{x_1}\underbrace{\boxed{\ 2\ }}_{x_2}\boxed{\overline{1}}\,\|\,\boxed{\overline{0}}\underbrace{\boxed{\ \ \overline{0}\ \ }}_{x_1+x_2-y_0} \end{array}$$

For $x_1 + x_2 - \overline{x}_1 > y_0$,

$$\begin{array}{l} \overbrace{\boxed{\quad\ 0 \quad\ }}^{y_0+\overline{x}_1}\overbrace{\boxed{\quad\ 1 \quad\ }}^{l-2y_0-\overline{x}_1}\overbrace{\boxed{\ \overline{0}\ }}^{y_0} \\ \underbrace{\boxed{\ 1\ }}_{x_1}\underbrace{\boxed{\quad 2 \quad}}_{x_2}\underbrace{\boxed{\quad \overline{0} \quad}}_{\overline{x}_1} \end{array}$$

The value of the energy function is 0.

A.2.2

Let $l - k \leq y_0$ and $2y_0 + k - l - x_1 + \overline{x}_1 > 0$. Then $b'_2 \otimes b'_1$ is depicted by

$$b'_2 \otimes b'_1 = \underbrace{\boxed{0}}_{y_0-l+k}\overbrace{\boxed{\ 1\ }}^{2l-k-2y_0}\underbrace{\boxed{\overline{0}}}_{y_0-l+k} \otimes \underbrace{\boxed{\quad 1 \quad}}_{x_1}\underbrace{\boxed{\quad 2 \quad}}_{x_2}\underbrace{\boxed{\qquad \overline{1} \qquad}}_{\overline{x}_1+l-k}$$

The column insertions $(b_2 \longrightarrow b_1)$ and $(b'_1 \longrightarrow b'_2)$ lead to the same intermediate result;
For $l - 2y_0 \geq 2\overline{x}_1$,

$$\underbrace{\boxed{\qquad 1 \qquad}}_{y_0-l+k} \longrightarrow \begin{array}{l} \overbrace{\boxed{\quad 0 \quad}}^{y_0+\overline{x}_1}\overbrace{\boxed{\qquad 1 \qquad}}^{l-2y_0-\overline{x}_1}\overbrace{\boxed{\quad \overline{0} \quad}}^{y_0} \\ \underbrace{\boxed{\quad 1 \quad}}_{l-y_0-\overline{x}_1-x_2}\underbrace{\boxed{\ \ 2\ \ }}_{x_2}\underbrace{\boxed{\quad \overline{0} \quad}}_{\overline{x}_1} \end{array}$$

For $l - 2y_0 < 2\overline{x}_1$,

$$\underbrace{\boxed{\qquad 1 \qquad}}_{y_0-l+k} \longrightarrow \begin{array}{l} \boxed{\quad 0 \quad}\overbrace{\boxed{0}}^{2\overline{x}_1-l+2y_0}\,\|\,\boxed{1}\boxed{\quad 1 \quad}\overbrace{\boxed{\quad \overline{0} \quad}}^{y_0} \\ \underbrace{\boxed{\ 1\ }}_{l-y_0-\overline{x}_1-x_2}\underbrace{\boxed{2}}_{x_2}\boxed{\overline{1}}\,\|\,\boxed{\overline{0}}\underbrace{\boxed{\quad \overline{0} \quad}}_{l-2y_0-\overline{x}_1} \end{array}$$

The value of the energy function is $y_0 - l + k$.

A.2.3

Let $y_0 < x_1 - \overline{x}_1$ and $2y_0 + k - l - x_1 + \overline{x}_1 \leq 0$. Then $b'_2 \otimes b'_1$ is depicted by

$$b'_2 \otimes b'_1 = \begin{array}{|c|c|c|}\hline \underset{y_0-w}{0} & \overset{k-2y_0+2w}{1} & \underset{y_0-w}{\overline{0}} \\ \hline \end{array} \otimes \begin{array}{|c|c|c|}\hline \overset{x_1+l-k-w}{1} & \underset{x_2}{2} & \underset{\overline{x}_1+w}{\overline{1}} \\ \hline \end{array}$$

where $w = (2y_0 - x_1 + \overline{x}_1)_+$. The column insertions $(b_2 \longrightarrow b_1)$ and $(b'_1 \longrightarrow b'_2)$ lead to the same intermediate result;

$$\begin{array}{|c|}\hline \underset{x_1-\overline{x}_1-y_0}{1} \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 0 & \overset{l-2y_0-\overline{x}_1}{1} & \overset{y_0}{\overline{0}} \\ \hline \underset{y_0+\overline{x}_1}{1} & \underset{x_2}{2} & \underset{\overline{x}_1}{\overline{0}} \\ \hline \end{array}$$

The value of the energy function is $x_1 - \overline{x}_1 - y_0$.

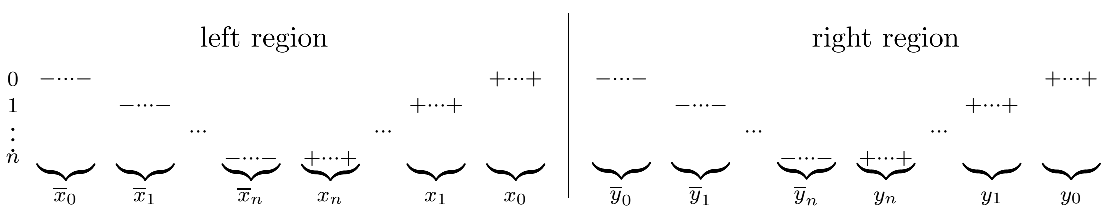
B Alternative rule for $C_n^{(1)}$

B.1 Algorithm for the isomorphism

Let $b_1 = (x_1, \dots, \overline{x}_1) \in B_l, b_2 = (y_1, \dots, \overline{y}_1) \in B_k$. We are going to show the rule of finding the image of $b_1 \otimes b_2$ under the isomorphism

$$\begin{aligned} \iota : B_l \otimes B_k &\overset{\sim}{\to} B_k \otimes B_l \\ b_1 \otimes b_2 &\mapsto b'_2 \otimes b'_1. \end{aligned}$$

Let $x_0 = \overline{x}_0 = (l - \sum_{i=1}^n (x_i + \overline{x}_i))/2$ and $y_0 = \overline{y}_0 = (k - \sum_{i=1}^n (y_i + \overline{y}_i))/2$. We assume $l \geq k$. We start with the following initial diagram.



By using Lemma 4.7 one can remove $(\boxed{0}, \boxed{\overline{0}})$ pairs from b_1 and b_2 simultaneously as many times as possible. Thus we assume in the following that either x_0 or y_0 is equal to 0 throughout. Then the general procedure to obtain the isomorphism and energy function is as follows.

0. Each symbol $+$ or $-$ is marked or unmarked. In the initial diagram all the symbols are unmarked.

1. There are three regions (left, right, and middle—the latter is empty in the initial diagram). Pick the leftmost symbol a in the right region. Find a's partner b in the left region according to the rule 2-3. Apply

(a), and repeat this procedure as many turns as possible, and then apply (b). During the procedure if a symbol named a is a $+$ (resp. $-$) symbol we call it $+_a$ (resp. $-_a$).

(a) If a exists and there is the partner b, mark b according to the rule 4. Put a new line on the right of a which forms the new boundary between the middle region and the right region. (In the second turn or later, delete the old line on the left of a.)

(b) If a does not exist or there is no partner of a, then stop. Enumerate the cardinality of the symbols in the right region and denote it by h. This h is equal to the value of the energy function, which is so normalized as the minimal value is equal to 0. Proceed to (c) or (d) according to the value of h.

(c) If $h = 0$, the procedure is finished. See (e).

(d) If $h > 0$, give up the diagram. Go back to the initial diagram and mark the leftmost h symbols in the left region. Then start again the procedure from the rule 1 in this new setting, and stop it keeping the rightmost h symbols in the right region untouched. Then see (e).

(e) The isomorphism ι is obtained as follows. At the end of the procedure, the marked symbols signify the contents of b_2', and the unmarked symbols signify the contents of b_1'.

2. If a is a $-$ symbol ($-_a$) in the i-th row, look at the i-th row in the left region.

   (a) If there are unmarked $+$ symbols in the i-th row in the left region, pick one of them and call it $+_c$.

      i. If there is no unmarked symbols (besides $+_c$) neither in the i-th row nor in the lower rows in the left region, then $+_c$ itself is identified with the partner $b(= +_b)$.

      ii. If there are unmarked symbols (besides $+_c$) either in the i-th row or in the lower rows in the left region, move $-_a$ and $+_c$ to the $(i-1)$-th row. Then apply the procedure (b).

   (b) If there is no unmarked $+$ symbol in the i-th row in the left region, or one has already done the procedure (a)-ii,

      i. If there are unmarked $-$ symbols in the left region whose positions are lower than that of $-_a$, then the partner $b(= -_b)$ is chosen from one of those $-$ symbols that has the highest position.

ii. If there is no unmarked $-$ symbol in the left region whose positions are lower than that of $-_a$, then the partner $b(=+_b)$ is chosen from one of the unmarked $+$ symbols in the left region that has the lowest position.

3. If a is a $+$ symbol $(+_a)$, then the partner $b(=+_b)$ is chosen from one of the unmarked $+$ symbols whose positions are higher than that of $+_a$ but the lowest among them.

4. If the partner b is a $-$ symbol $(-_b)$, mark it. If b is a $+$ symbol $(+_b)$ in the j-th row, look at the j-th row in the left and the middle region.

   (a) If there are unmarked $-$ symbols in the j-th row either in the left region or in the middle region, pick the leftmost one of them and call it $-_d$. Move $+_b$ and $-_d$ to the $(j+1)$-th row and then mark the $+_b$.

   (b) If there is no unmarked $-$ symbol in the j-th row neither in the left region nor in the middle region, then mark the $+_b$.

This description of the rule is derived from the column insertion rule in Section 3.2 accompanied with the *reverse row insertion procedure* for the C-tableaux. We do not describe the latter procedure in this paper.

In the rule 4-(a), we have chosen $-_d$ to be the leftmost one. However the final result of the procedure is actually the same for any choice of the $-$ symbols in the j-th row of the middle and left regions.

B.2 Examples

Let us present two examples. We signify the marked symbols with circles.

B.2.1 Example 1

Let us derive

$$\iota : 1134\bar{3}\bar{2}\bar{1} \otimes \bar{4}\bar{4}\bar{4}\bar{1}\bar{1} \mapsto 144\bar{2}\bar{1} \otimes 0\bar{4}\bar{4}\bar{4}\bar{4}\bar{1}\bar{0} \tag{B.1}$$

under the isomorphism $B_7 \otimes B_5 \simeq B_5 \otimes B_7$ of the $U_q'(C_4^{(1)})$ crystals. The value of the energy function is 0 for this element. The initial diagram is as follows.

$$\begin{array}{c|ccccccc|ccccc}
0 & & & & & & & & & & & & \\
1 & - & & & & & + & + & - & - & & & \\
2 & & - & & & & & & & & & & \\
3 & & & - & & + & & & & & & & \\
4 & & & & + & & & & & & - & - & -
\end{array}$$

We apply 2-(a)-ii, and then 2-(b)-i.

$$\begin{array}{rccccccc|c|cccc}
0 & & & & & & & + & - & & & & \\
1 & \ominus & & & & & + & & & - & & & \\
2 & & - & & & & & & & & & & \\
3 & & & - & & + & & & & & & & \\
4 & & & & + & & & & & & - & - & -
\end{array}$$

Again, we apply 2-(a)-ii, and then 2-(b)-i.

$$\begin{array}{rccccccc|cc|ccc}
0 & & & & & & + & + & - & - & & & \\
1 & \ominus & & & & & & & & & & & \\
2 & & \ominus & & & & & & & & & & \\
3 & & & - & & + & & & & & & & \\
4 & & & & + & & & & & & - & - & -
\end{array}$$

We apply 2-(a)-i.

$$\begin{array}{rccccccc|ccc|cc}
0 & & & & & & + & + & - & - & & & \\
1 & \ominus & & & & & & & & & & & \\
2 & & \ominus & & & & & & & & & & \\
3 & & & - & & + & & & & & & & \\
4 & & & & \oplus & & & & & & - & - & -
\end{array}$$

We apply 2-(b)-ii to find the partner, and then apply 4-(a) to mark the partner.

$$\begin{array}{rccccccc|cccc|c}
0 & & & & & & + & + & - & - & & & \\
1 & \ominus & & & & & & & & & & & \\
2 & & \ominus & & & & & & & & & & \\
3 & & & & & & & & & & & & \\
4 & & & - & \oplus & \oplus & & & & & - & - & -
\end{array}$$

Again we apply 2-(b)-ii to find the partner, and then apply 4-(a) to mark the partner.

$$\begin{array}{rccccccc|ccccc|}
0 & & & & & & & + & - & & & & \\
1 & \ominus & & & & & \oplus & & & - & & & \\
2 & & \ominus & & & & & & & & & & \\
3 & & & & & & & & & & & & \\
4 & & & - & \oplus & \oplus & & & & & - & - & -
\end{array}$$

The procedure is finished. Here the set of marked symbols stands for $144\bar{2}\bar{1} \in B_5$, and the set of unmarked symbols stands for $0\bar{4}\bar{4}\bar{4}\bar{4}\bar{1}\bar{0} \in B_7$.

Let us check the isomorphism by the definition.

$$\begin{array}{ccc}
1134\bar{3}\bar{2}\bar{1} \otimes \bar{4}\bar{4}\bar{4}\bar{1}\bar{1} & \xrightarrow{\iota} & 144\bar{2}\bar{1} \otimes 0\bar{4}\bar{4}\bar{4}\bar{4}\bar{1}\bar{0} \\
{\scriptstyle\psi_1}\big\downarrow & & {\scriptstyle\psi_1}\big\downarrow \\
1111111 \otimes 122\bar{1}\bar{1} & \xrightarrow{\iota} & 11111 \otimes 01122\bar{1}\bar{0}
\end{array} \tag{B.2}$$

where

$$\psi_1 = (\tilde{e}_2)^2(\tilde{e}_3)^2(\tilde{e}_1)^6(\tilde{e}_2)^6(\tilde{e}_3)^4(\tilde{e}_4)^6(\tilde{e}_3)^4(\tilde{e}_2)^2\tilde{e}_1. \tag{B.3}$$

We arrive at a $U_q(C_4)$ highest element.

$$\begin{array}{ccc} 1111111 \otimes 122\bar{1}\bar{1} & \xrightarrow{\iota} & 11111 \otimes 01122\bar{1}\bar{0} \\ \psi_2 \downarrow & & \psi_2 \downarrow \\ 1111111 \otimes 11111 & \xrightarrow{\iota} & 11111 \otimes 1111111 \end{array} \tag{B.4}$$

where

$$\psi_2 = (\tilde{f}_0)^8 (\tilde{f}_1)^3 (\tilde{f}_2)^3 (\tilde{f}_3)^3 (\tilde{f}_4)^3 (\tilde{f}_3)^3 (\tilde{f}_2)^3 \tilde{f}_1 (\tilde{e}_0)^3. \tag{B.5}$$

The energy was raised by 1 when the third $\tilde{e}_0$ was applied, and lowered by 1 when the first $\tilde{f}_0$ was applied. Then it was raised by 5 when the fourth to the eighth $\tilde{f}_0$'s were applied.

B.2.2 Example 2

Let us derive

$$\iota : 0\bar{2}\bar{2}\bar{1}\bar{1}\bar{1}\bar{0} \otimes 1112\bar{2}\bar{1} \mapsto 0\bar{2}\bar{1}\bar{1}\bar{1}\bar{0} \otimes 1112\bar{2}\bar{2}\bar{1} \tag{B.6}$$

under the isomorphism $B_7 \otimes B_6 \simeq B_6 \otimes B_7$ of the $U'_q(C_2^{(1)})$ crystals. The value of the energy function is 4 for this element. The initial diagram is as follows.

$$\begin{array}{llllll|lll} 0 & - & & & & + & & & \\ 1 & & - - - & & & & - & & + + + \\ 2 & & & - - & & & & - + & \end{array}$$

We apply 2-(b)-i.

$$\begin{array}{llllll|l|ll} 0 & - & & & & + & & & \\ 1 & & - - - & & & & - & & + + + \\ 2 & & & - \ominus & & & & - + & \end{array}$$

We apply 2-(b)-ii to find the partner, and then apply 4-(a) to mark the partner.

$$\begin{array}{lllll|ll|ll} 0 & & & & & & & & \\ 1 & - - - - & & & \oplus & - & & & + + + \\ 2 & & & - \ominus & & & - & + & \end{array}$$

This time we find that there is no partner in the left region for the leftmost + symbol in the right region. We interrupt the procedure here according to 1-(b). Since there are four symbols in the right region, we find that the value of the energy function is equal to 4. Following 1-(d) we give up this diagram and go to the initial diagram with four marked symbols.

$$\begin{array}{llllll|lll} 0 & \ominus & & & & + & & & \\ 1 & & \ominus\ominus\ominus & & & & - & & + + + \\ 2 & & & - - & & & & - + & \end{array}$$

We apply 2-(b)-i.

$$\begin{array}{ll|l|l}
0 & \ominus \qquad\qquad\qquad + & & \\
1 & \quad \ominus\ominus\ominus & - & \quad +++ \\
2 & \qquad\qquad -\ominus & & - + \\
\end{array}$$

We apply 2-(b)-ii to find the partner, and then apply 4-(b) to mark the partner.

$$\begin{array}{ll|l|l}
0 & \ominus \qquad\qquad\qquad \oplus & & \\
1 & \quad \ominus\ominus\ominus & - & \quad +++ \\
2 & \qquad\qquad -\ominus & \quad - & + \\
\end{array}$$

The procedure is finished. Here the set of marked symbols stands for $0\bar{2}\bar{1}\bar{1}\bar{1}\bar{0} \in B_6$, and the set of unmarked symbols stands for $1112\bar{2}\bar{2}\bar{1} \in B_7$.

C $C_n^{(1)}$ Kostka polynomials

Let $\mu_1 \geq \cdots \geq \mu_L (\geq 1)$ be a set of integers. We set $\mu = (\mu_1, \mu_2, \ldots, \mu_L)$. Consider the tensor product of $U_q'(C_n^{(1)})$ crystals $B_{\mu_1} \otimes \cdots \otimes B_{\mu_L}$. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be another partition satisfying $|\lambda| \leq |\mu|$ and $|\lambda| \equiv |\mu| \pmod 2$. We define a classically restricted 1dsum (cf. [HKOTY]):

$$X_{\lambda,\mu}(t) = \sum\nolimits^{*} t^{\sum_{0 \leq i < j \leq L} H(b_i \otimes b_j^{(i+1)})}, \tag{C.1}$$

where the sum $\sum^*$ is taken over all $b_1 \otimes \cdots \otimes b_L \in B_{\mu_1} \otimes \cdots \otimes B_{\mu_L}$ satisfying

$$\tilde{e}_i(b_1 \otimes \cdots \otimes b_L) = 0,\ \varphi_i(b_1 \otimes \cdots \otimes b_L) = \lambda_i - \lambda_{i+1}, \quad 1 \leq i \leq n\ (\lambda_{n+1} = 0).$$

This condition is equivalent to

$$(b_L \xrightarrow{*} \cdots \xrightarrow{*} (b_3 \xrightarrow{*} (b_2 \xrightarrow{*} b_1)) \cdots) = T(\lambda). \tag{C.2}$$

$T(\lambda)$ is the unique tableau of both shape and weight λ. Namely all the letters in the first row are 1 and those in the second row are 2, and so on. (We distinguished b_j from $\mathbb{T}(b_j)$.) In the summation we set $b_i^{(i)} = b_i$, and $b_j^{(i)}$ $(i < j)$ are defined by successive use of the crystal isomorphism,

$$\begin{array}{ccccc}
B_{\mu_i} \otimes \cdots \otimes B_{\mu_{j-1}} \otimes B_{\mu_j} & \simeq & B_{\mu_i} \otimes \cdots \otimes B_{\mu_j} \otimes B_{\mu_{j-1}} & \simeq & \cdots \\
b_i \otimes \cdots \otimes b_{j-1} \otimes b_j & \mapsto & b_i \otimes \cdots \otimes b_j^{(j-1)} \otimes b'_{j-1} & \mapsto & \cdots \\
 & & \cdots \quad \simeq \quad B_{\mu_j} \otimes B_{\mu_i} \otimes \cdots \otimes B_{\mu_{j-1}} & & \\
 & & \cdots \quad \mapsto \quad b_j^{(i)} \otimes b'_i \otimes \cdots \otimes b'_{j-1}. & &
\end{array}$$

The above condition (C.2) implies that $b_j^{(1)}$ in the tableau presentation should have the form $\mathbb{T}(b_j^{(1)}) = \boxed{0 \cdots 0}\boxed{1 \cdots 1}\boxed{\bar{0} \cdots \bar{0}}$. b_0 is chosen so that $H(b_0 \otimes$

$b_j^{(1)}) = -\sharp(\boxed{0}\,\mathrm{in}\,\mathbb{T}(b_j^{(1)}))$. Up to additive constant this agrees with the choice of b_0 in [HKOTY]. Up to an overall power of t, this is a polynomial which may be viewed as a $C_n^{(1)}$-analogue of the Kostka polynomial. In fact if $|\lambda| = |\mu|$, $X_{\lambda,\mu}(t)$ coincides with the ordinary Kostka polynomial $K_{\lambda,\mu}(t)$.

Following the tables in pp. 239-240 of [Ma] we give a list of $X_{\lambda,\mu}(t)$ or the matrices $X(t) := \{X_{\lambda,\mu}(t)\}$ for $|\mu| \le 6$ and $|\lambda| = |\mu| - 2, |\mu| - 4, \ldots$ with $n \ge L$. $X_{\lambda,\mu}(t)$ is independent of n if $n \ge L$. If $n < L$ it is n-dependent in general. For instance, let $\lambda = (1^3)$ and $\mu = (1^5)$. The element $\boxed{1} \otimes \boxed{2} \otimes \boxed{3} \otimes \boxed{4} \otimes \boxed{\overline{4}} \in (B_1)^{\otimes 5}$ contributes to $X_{(1^3),(1^5)}(t)$ for $n \ge 4$, but does not for $n = 2, 3$. We have checked that all the data in the table agrees with the fermionic formula in [HKOTY]. In the tables, a row (resp. column) specifies λ (resp. μ) in $X_{\lambda,\mu}(t)$.

$$X_{\emptyset,(2)}(t) = t^{-1}, \quad X_{\emptyset,(1^2)}(t) = 1.$$

$$X_{(1),(3)}(t) = t^{-1}, \quad X_{(1),(21)}(t) = t^{-1} + 1, \quad X_{(1),(1^3)}(t) = 1 + t + t^2.$$

	(4)	(31)	(2^2)	(21^2)	(1^4)
$\emptyset$	t^{-2}	t^{-1}	$t^{-2}+1$	$t^{-1}+t$	$1+t^2+t^4$
(2)	t^{-1}	$t^{-1}+1$	$t^{-1}+1+t$	$2+t+t^2$	$t+t^2+2t^3+t^4+t^5$
(1^2)		t^{-1}	1	$t^{-1}+1+t$	$1+t+2t^2+t^3+t^4$

	(5)	(41)	(32)	(31^2)	(2^21)
(1)	t^{-2}	$t^{-2}+t^{-1}$	$t^{-2}+t^{-1}+1$	$2t^{-1}+1+t$	$t^{-2}+t^{-1}+2+t+t^2$
(3)	t^{-1}	$t^{-1}+1$	$t^{-1}+1+t$	$2+t+t^2$	$1+2t+t^2+t^3$
(21)		t^{-1}	$t^{-1}+1$	$t^{-1}+2+t$	$t^{-1}+2+2t+t^2$
(1^3)				t^{-1}	1

	(21^3)	(1^5)
(1)	$t^{-1}+2+2t+2t^2+t^3+t^4$	$1+t+2t^2+2t^3+3t^4+2t^5+2t^6+t^7+t^8$
(3)	$t+2t^2+2t^3+t^4+t^5$	$t^3+t^4+2t^5+2t^6+2t^7+t^8+t^9$
(21)	$2+3t+3t^2+2t^3+t^4$	$t+2t^2+3t^3+4t^4+4t^5+3t^6+2t^7+t^8$
(1^3)	$t^{-1}+1+t+t^2$	$1+t+2t^2+2t^3+2t^4+t^5+t^6$

	(6)	(51)	(42)	(41^2)	(3^2)	(321)
$\emptyset$	t^{-3}	t^{-2}	$t^{-3}+t^{-1}$	$t^{-2}+1$	$t^{-2}+1$	$t^{-2}+t^{-1}+t$
(2)	t^{-2}	$t^{-2}+t^{-1}$	$2t^{-2}+t^{-1}+1$	$3t^{-1}+1+t$	$2t^{-1}+1+t$	$t^{-2}+2t^{-1}+3+t+t^2$
(1^2)		t^{-2}	t^{-1}	$t^{-2}+t^{-1}+1$	$t^{-2}+1$	$t^{-2}+2t^{-1}+1+t$
(4)	t^{-1}	$t^{-1}+1$	$t^{-1}+1+t$	$2+t+t^2$	$1+t+t^2$	$1+2t+t^2+t^3$
(31)		t^{-1}	$t^{-1}+1$	$t^{-1}+2+t$	$t^{-1}+1+t$	$t^{-1}+3+2t+t^2$
(21^2)				t^{-1}		$t^{-1}+1$
(2^2)			t^{-1}	1	1	$t^{-1}+1+t$
(1^4)						

	(31^3)	(2^3)
$\emptyset$	$t^{-1}+1+t+t^3$	$t^{-3}+t^{-1}+1+t+t^3$
(2)	$t^{-1}+3+3t+3t^2+t^3+t^4$	$t^{-2}+t^{-1}+3+2t+3t^2+t^3+t^4$
(1^2)	$2t^{-1}+2+3t+t^2+t^3$	$t^{-1}+1+2t+t^2+t^3$
(4)	$t+2t^2+2t^3+t^4+t^5$	$t+t^2+2t^3+t^4+t^5$
(31)	$2+3t+4t^2+2t^3+t^4$	$1+2t+3t^2+2t^3+t^4$
(21^2)	$t^{-1}+2+2t+t^2$	$1+t+t^2$
(2^2)	$1+2t+t^2+t^3$	$t^{-1}+1+2t+t^2+t^3$
(1^4)	t^{-1}	

	(2^21^2)	(21^4)
$\emptyset$	$t^{-2}+2+t+t^2+t^4$	$t^{-1}+2t+t^2+2t^3+t^4+t^5+t^7$
(2)	$2t^{-1}+2+5t+3t^2+3t^3+t^4+t^5$	$2+2t+5t^2+4t^3+6t^4+3t^5+3t^6+t^7+t^8$
(1^2)	$t^{-2}+t^{-1}+4+2t+3t^2+t^3+t^4$	$t^{-1}+2+4t+4t^2+5t^3+3t^4+3t^5+t^6+t^7$
(4)	$2t^2+2t^3+2t^4+t^5+t^6$	$t^3+t^4+3t^5+2t^6+2t^7+t^8+t^9$
(31)	$1+4t+4t^2+4t^3+2t^4+t^5$	$t+3t^2+5t^3+6t^4+5t^5+4t^6+2t^7+t^8$
(21^2)	$t^{-1}+2+3t+2t^2+t^3$	$2+3t+5t^2+4t^3+4t^4+2t^5+t^6$
(2^2)	$2+2t+3t^2+t^3+t^4$	$2t+2t^2+4t^3+3t^4+3t^5+t^6+t^7$
(1^4)	1	$t^{-1}+1+t+t^2+t^3$

	(1^6)
$\emptyset$	$1+t^2+t^3+2t^4+t^5+3t^6+t^7+2t^8+t^9+t^{10}+t^{12}$
(2)	$t+t^2+3t^3+3t^4+6t^5+5t^6+7t^7+5t^8+6t^9+3t^{10}+3t^{11}+t^{12}+t^{13}$
(1^2)	$1+t+3t^2+3t^3+6t^4+5t^5+7t^6+5t^7+6t^8+3t^9+3t^{10}+t^{11}+t^{12}$
(4)	$t^6+t^7+2t^8+2t^9+3t^{10}+2t^{11}+2t^{12}+t^{13}+t^{14}$
(31)	$t^3+2t^4+4t^5+5t^6+7t^7+7t^8+7t^9+5t^{10}+4t^{11}+2t^{12}+t^{13}$
(21^2)	$t+2t^2+4t^3+5t^4+7t^5+7t^6+7t^7+5t^8+4t^9+2t^{10}+t^{11}$
(2^2)	$t^2+t^3+3t^4+3t^5+5t^6+4t^7+5t^8+3t^9+3t^{10}+t^{11}+t^{12}$
(1^4)	$1+t+2t^2+2t^3+3t^4+2t^5+2t^6+t^7+t^8$

References

[ABF] G. E. Andrews, R. J. Baxter and P. J. Forrester, *Eight vertex SOS model and generalized Rogers-Ramanujan-type identities*, J. Stat. Phys. **35**, (1984) 193-266.

[B] T.H. Baker, *An insertion scheme for C_n crystals*, RIMS preprint 1234 (1999).

[Ba] R.J. Baxter, *Exactly solved models in statistical mechanics*, Academic Press, London (1982).

[Be] H. A. Bethe, Zur Theorie der Metalle, I. Eigenwerte und Eigenfunktionen der linearen Atomkette, Z. Physik **71** (1931) 205–231.

[DJKMO] E. Date, M. Jimbo, A. Kuniba, T. Miwa and M. Okado, One dimensional configuration sums in vertex models and affine Lie algebra characters, Lett. Math. Phys. **17** (1989) 69–77.

[F] W. Fulton, *Young tableaux: with applications to representation theory and geometry*, London Math. Soc. student texts **35**, Cambridge University Press (1997).

[FOY] K. Fukuda, M. Okado and Y. Yamada, *Energy functions in box ball systems*, preprint math.QA/9908116.

[HHIKTT] G. Hatayama, K. Hikami, R. Inoue, A. Kuniba, T. Takagi and T. Tokihiro, in preparation.

[HKKOT] G. Hatayama, Y. Koga, A. Kuniba, M. Okado and T. Takagi, *Finite crystals and paths*, preprint, math.QA/9901082.

[HKKOTY] G. Hatayama, A. N. Kirillov, A. Kuniba, M. Okado, T. Takagi and Y. Yamada, Character formulae of $\widehat{sl}_n$-modules and inhomogeneous paths, Nucl. Phys. **B536** [PM] (1999) 575-616.

[HKOTY] G. Hatayama, A. Kuniba, M. Okado, T. Takagi and Y. Yamada, *Remarks on Fermionic Formula*, preprint, math.QA/9812022.

[HKT] G. Hatayama, A. Kuniba and T. Takagi, *Soliton cellular automata associated with finite crystals*, preprint solv-int/9907020.

[KKM] S-J. Kang, M. Kashiwara and K. C. Misra, Crystal bases of Verma modules for quantum affine Lie algebras, RIMS preprint 887 (1992); Compositio Math. **92** (1994) 299-325.

[KMN1] S-J. Kang, M. Kashiwara, K. C. Misra, T. Miwa, T. Nakashima and A. Nakayashiki, Affine crystals and vertex models, Int. J. Mod. Phys. A **7** (suppl. 1A) (1992) 449-484.

[KMN2] S-J. Kang, M. Kashiwara, K. C. Misra, T. Miwa, T. Nakashima and A. Nakayashiki, Perfect crystals of quantum affine Lie algebras, Duke Math. J. **68** (1992) 499-607.

[KN] M. Kashiwara and T. Nakashima, Crystal graph for representations of the q-analogue of classical Lie algebras, J. Alg. **165** (1994) 295-345.

[KR] A. N. Kirillov and N. Yu. Reshetikhin, *The Bethe ansatz and the combinatorics of Young tableaux*, J. Sov. Math. **41** (1988) 925-955.

[Ma] I. Macdonald, *Symmetric functions and Hall polynomials*, 2nd edition, Oxford Univ. Press, New York (1995).

[NY] A. Nakayashiki and Y. Yamada, Kostka polynomials and energy functions in solvable lattice models, Selecta Mathematica, New Ser. **3** (1997) 547-599.

[S] M. Shimozono, *Affine Type A Crystal Structure on Tensor Product of Rectangles, Demazure Characters, and Nilpotent Varieties*, preprint math.QA/9804039.

[SW] A. Schilling and S. O. Warnaar, *Inhomogeneous lattice paths, generalized Kostka polynomials and A_{n-1} supernomials*, Commun. Math. Phys. **202** (1999) 359-401.

[TS] D. Takahashi and J. Satsuma, *A soliton cellular automaton*, J. Phys. Soc. Jpn. **59** (1990) 3514–3519.

[TTMS] T. Tokihiro, D. Takahashi, J. Matsukidaira and J. Satsuma, *From soliton equations to integrable cellular automata through a limiting procedure*, Phys. Rev. Lett. **76** (1996) 3247–3250.